# **University of Reading**

**School of Mathematics, Meteorology & Physics** 

## **EFFICIENT EVALUATION OF HIGHLY OSCILLATORY INTEGRALS**

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This dissertation is submitted to the Department of Mathematics in partial fulfilment of the requirements for the degree of Master of Science

#### **Declaration**

**I confirm that this is my own work, and the use of all material from other sources has been properly and fully acknowledged.** 

Shaun Potticary

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# **Contents**



# **Integral 2: Irregular Oscillators**

## **1.) Introduction**

 Highly oscillatory integrals appear in many types of mathematical problems including wave scattering problems, quantum chemistry, electrodynamics and fluid dynamics and Fourier transforms. Although these integrals appear in many places, in this project we are going to concentrate on the wave scattering problem and the Fourier transforms. A Fourier transform is used when solving two dimensional partial differential equations, and in the evaluation of a complex Fourier series. In both cases the evaluation of the Fourier transform involves the evaluation of an integral that may be highly oscillatory, (commonly it is so). The question is how do we compute this? A Fourier transform is of the form

$$
\frac{1}{\sqrt{2}} \qquad f(t)e^{-ikt} dt \qquad (1.1)
$$

The reason that the evaluation of these integrals is so important is because standard methods to evaluate complex integrals don't work well in the case that the integrand oscillates rapidly. This is why in this project specific techniques will be looked at for the problem and we will see how good at approximating (1.1) they are. Firstly the integrals that we are going to look at need to be defined. Two forms of highly oscillatory integrals are going to be looked at. The first one is a good place to start with our investigation.

**Definition 1.1 - The general form our first integral that is dealt with in this project is:-**

$$
I[f]: \int_{0}^{b} f(x)e^{ikx} dx
$$
 (1.2)

where  $f(x)$  is a smooth and slowly oscillating function. This integral is actually included in the second integral, but the techniques that we are going to use in this project actually are a lot simpler for this one. The form of I[f] is a general form of a complex Fourier series and this is going to be our motivation of the first part of the project.

**Definition 1.2** - The second integral that we are going to look at is:-

$$
J[f]: \int_{a}^{b} f(x)e^{ikg(x)}dx
$$
 (1.2)

(namely the midpoint rule and the trapezium rule) so as to simplify the analysis. We do know that Gaussian quadrature gives better results in practice, though the analysis is too complicated to get good theoretical results beyond those already achieved in  $(1).$ 

We should expect from this integral to get an O(1/k) for the integral which is what the analytical solution would give, and so for any method to converge quickly we would expect the method to have a similar order.

 For the second integral we will look at two types of methods. The first one is an asymptotic type method called method of stationary phase (MOSP) which is different for g'(x)=0 in [a,b] than it is for g'(x) 0 in [a,b]. The analysis for this part will not be in great detail, only the procedure is mainly shown in this section as it is this that is needed for the second part. The second method is a Filon type quadrature method used for the first integral but again it has the problem if  $g'(x)=0$  in [a,b] then a fix has to be used to get a solution. The quadrature method that is going to be used with

## **Integral 1: Fourier Transforms**

In this section we will deal with the first integral that we are interested in I[f], as described in definition 1.1

## **2.) Quadrature Methods**

#### **Midpoint Rule and the Composite Midpoint Rule**

 The Midpoint Rule is the easiest but also one of the least accurate of all the quadrature methods that we are going to look at in this project. The best way of looking at the Midpoint Rule is to see what the formula is on a general function  $g(x)$ , and then look at the error approximation for that. Once we have done this then we can look at the Midpoint Rule when our function is in fact our oscillatory integral I[f]. After we have looked at the Midpoint Rule we can look at what happens when we split up the interval [a,b] into N separate intervals and then use the Midpoint Rule on each interval separately, this is called the Composite Midpoint Rule.

**Definition 2.1** – The Midpoint Rule applied to a general function g(x), between the limits [a,b] is given below.

$$
\int_{a}^{b} g(x)dx \quad (b-a)g(\frac{a-b}{2})
$$
 (2.1)

The Midpoint Rule finds the midpoint of the interval  $(\frac{a-b}{2})$  and calculates g at this point, and then this all has to be multiplied by the length of our interval, (b-a). In fact the Midpoint Rule actually approximates the function  $q(x)$  by a constant and then finds the area under the resulting rectangle. This means that our function  $q(x)$  is a constant then it will solve the integral exactly. As a consequence the midpoint rule actually solves exactly for a polynomial of degree 1, which is because even though the midpoint rule doesn't approximate the line exactly, the integral is exact.

The diagram above illustrates the Midpoint Rule for a general func

Now when we calculate  $|I - I_m|$ 

e first I

this term can be simplified to get one term, this is done by noticing that the modulus of the exponential term is equal to 1. The other term can also be simplified by splitting the term up using the triangular inequality giving us

 $|k|f()|$ 

and adds them up to give an approximation to the true integral. The figure below shows an example of how the composite midpoint rule works. The function  $g(x)$  is a general function and we have 4 intervals in [a,b] for which the function is going to be approximated on. As can be seen the midpoint is shown and a horizontal line is drawn in each section. It is clear that the midpoint rule is not very good at approximating an

```
N a jh
            g(x)dx The exact solution I[f]
\mathbf{L}j 1 a (j i)h
       N a jh
               g(a \t (j \t 1/2)h)dx The composite midpoint approximation to g(x)\Gamma_{\rm cm}j 1 a (j 1)h
```
we work through the error analysis in the same way as we did for the original Midpoint rule. We replace the  $g(x)$  in I by a Taylor series applied at  $a+(j-1/2)$ h to give

<sub>om</sub> │ │ │ │ │ │ (x (a (j 1/2)h))g'(a (j 1/2)h)dx │ (x (a (j 1/2)h))<sup>2</sup>  $\frac{g''(\ )}{2!}$ dx  $N$  a jh j 1 a (j 1)h <sup>a jh</sup> (x (a (j 1/2)h))<sup>2</sup>  $\frac{g''()}{2!}dx$  | a (j 1)h

where once again [a,b]. Now we can integrate, but noticing that the limits have changed from when we did this before, doesn't actually make a difference to the first term, this cancels as before leaving us only the second term and also noticing that  $(a +ih) - (a + (i-1)h) = h$  we get an error term

$$
|e_{cm}| \int_{j_1}^{N} |g''( )| \frac{h^3}{24} \frac{Nh^3}{24} |g''( )| Ch^2 |g''( )|
$$

 This is therefore the error as was given in (2.9) and the power of h has been decreased because we have summed up all of the intervals and C=1/24.

**Definition 2.3** - Now we need a formula for the midpoint rule applied to I[f], and this is produced by just replacing  $g(x)$  by  $f(x)e^{ikx}$  to give

$$
\begin{array}{ccc}\n\bullet & 4 & & \bullet \\
\bullet & & & \bullet\n\end{array}
$$

interested in we see that the error term grows so our step size has to be very small for us to get a good result.

 **Lemma 2.1** - For our composite midpoint approximation to I[f] it can be easier to write the error in the form of the greatest factors for which the error is dependant, this form is

$$
e_i = C h k^2
$$

 So it is clear that if k is large like we are intending looking to solve for, then the error is going to grow at a rate of k squared. The only way to decrease the error is to make h very small. The second form for our error is the best to see what exactly we need to do to get a good result, (this of course is if all other terms are relatively small). In this case as k gets large we need to let h get small at the same rate and as the composite midpoint rule is a poor approximation on a non-oscillatory integral on an oscillatory integral it is even worse and h has to be very small for us to get a good approximation. We now need to be able to see some practical results of the composite midpoint rule, and to do this a program in Matlab has been created to find the error for a given function.

**Example 2.1** - The function that we are going to look at is

$$
e^{(1 + ik)x}dx
$$
 (2.12)

This is our integral I[f], where here  $f(x) = e^{x}$ .

 The program has been run to show how good the method is on a non oscillatory integral (when k is small), mildly oscillatory integral (when k is 10) and a highly oscillatory integral (when k is 100). The program calculates the approximate value, the exact value and the absolute error that is found. In the table below the program has been run with k=1, 10 and 100 for the number of step sizes being 4, 8, 16, 32, 64, and 128. We can approximately estimate the error that we should be seeing for our example. This is

$$
e_j
$$
 C(hk)<sup>2</sup>|e<sup>(1 ik)</sup> |  
C(hk)<sup>2</sup> [0,1].

 The third term is restricted and is included in the constant so we are looking at the error to be dependant on h and k. We should expect to get reasonable results when k is small, but not anything that will converge fast, and as k gets larger we should expect

to see that the results do get worse and it takes many intervals N to get the error even under 1.

k	N	Error $e_{cm}$
1	4	8.598644053E-03
	8	2.148751918E-03
	16	5.374381485E-04
	32	1.343599540E-04
	64	3.35898850E-05
	128	8.3974712E-06
10	4	1.143894736E-01
	8	2.465250208E-02
	16	5.953557152E-03
	32	1.475782821E-03
	64	4.368165183E-04
	128	9.199263084E-05
100	$\overline{\mathcal{A}}$	1.704945960
	8	1.707292228
	16	1.690171991
	32	1.082230203E-02
	64	2.105996229E-03
	128	4.981234001E-04

**Table 2.1** 

 The table above shows our results, it is clear that when k is small the midpoint needs about 16 intervals to get a really good result, but as k get bigger we see that the number of intervals needs to increases to get the same error. The next table shows about how many intervals, N, we need to get a 1% relative error. This is the error that we have divided by the exact solution

**Table 2.2** 

		20	40	80	160	320	640
	64	129	258	516	1033	2066	4132

 The table above gives a very interesting result, this is that as k doubles then the number that N has to increase by to get a 1% relative error is also double.

### **Trapezium rule and the Composite Trapezium Rule**

 The next quadrature method that we are going to look at is the Trapezium Rule. We would expect the Trapezium Rule to be better than the Midpoint Rule as it uses more points to get an approximation, but depending on the function it is applied on, the Trapezium Rule is pretty much the same as the Midpoint Rule. The Trapezium Rule calculates the function at the two endpoints and calculates the area under the trapezium formed.

**Definition 2.4-** The Trapezium rule when applied to a general function  $q(x)$  between the limits [a,b], is

$$
\int_{a}^{b} g(x) \quad (\frac{1}{2}f(a) \quad \frac{1}{2}f(b))(b \quad a) \tag{2.13}
$$

 The formula above is one way to look at the Trapezium Rule, but another way to look at it is that we wish to draw a line between f(a) and f(b), and then integrate under that line. The reason we wish to do this is because the error analysis is easier when looking at the Trapezium Rule in this way so from now on in this project the Trapezium Rule will be dealt with in the form of Definition 2.5.

**Definition 2.5-** The trapezium rule when applied to  $q(x)$  in [a,b] can be written as

$$
g(x) = \int_{a}^{b} g(x) \int_{a}^{b} (cx + d)dx
$$
  
where  $\frac{g(b) - g(a)}{b-a}$  and  $d = \frac{bg(a) - ag(b)}{b-a}$  (2.14)

#### **Proof**

We wish to create a line that at  $x=a$  is  $g(a)$  and at  $x=b$  is  $g(b)$ , so therefore we need to create two simultaneous equation of the form

- 1)  $ca + d = g(a)$
- 2)  $cb + d = g(b)$

Solving these equations for c and d gives us our formula.

 The diagram below is a good example of how the trapezium rule works on a general function  $q(x)$ 



The best way to see how good at approximating  $g(x)$  the Trapezium Rule is is to look at the error analysis.

**Theorem 2.4-** The error term for the Trapezium Rule when applied to a general  $g(x)$ in [a,b] is

$$
e_t \quad \frac{(b \quad a)^3}{12} |g''( )| \tag{2.15}
$$

and when applied to I[f] the error term becomes

$$
e_m \quad \frac{(b-a)^3}{12}(k)^2 |f''(x)| \tag{2.16}
$$

#### **Proof**

To work out the error we need to find  $|e|=|I_1|$  where

\n
$$
\int_{a}^{b} g(x) \, dx
$$
\n The exact solution  $I[f]$ \n

\n\n $\int_{t}^{a} (cx \, d) \, dx$ \n The trapezium approximation to  $g(x)$ ,  $c$ , and  $d$  are as defined earlier.\n

so our error term is of the form

$$
|e_t| \mid \int_a^b [g(x) \quad cx \quad d] dx \mid \tag{2.17}
$$

 To get this term into a better form and to be able to find out the leading order of the error we need to use Theorem 2.5

**Theorem 2.5 -** From "Burden & Faires- Numerical analysis" p111 (2).

Suppose  $x_0, x_1, \ldots, x_n$  are distinct numbers in [a,b]. Then for each x in [a,b], a number in (a,b) exists with

$$
f(x)
$$
 P(x)  $\frac{f^{(n-1)}(x)}{(n-1)!}(x-x_0)(x-x_1)......(x-x_n)$  (2.18)

where  $P(x)$  is a polynomial of degree n, which is the polynomial that interpolates f at the points  $x_0, x_1, \ldots, x_n$ .

We can use this theorem in our error analysis, letting  $P(x) = cx + d$  and n=1, and setting  $x_0$ =a,  $x_1$ =b be the points in (2.18). then

g(x) cx d 
$$
\frac{g''()}{2}(x \ a)(x \ b)
$$
 (2.19)

and we can substitute this into (2.13) to give us our new error estimate

$$
e_t \int_a^b \frac{g''(x)}{2} (x + a)(x + b)] dx
$$

 Now we can once again just integrate the expression above and input the limits to give us our error term.

$$
e_{t} \quad | \frac{g''(\ )}{2}^{b} \quad (x^{2} \quad (a \quad b)x \quad ab)dx \mid | \frac{g''(\ )}{2} [\frac{x^{3}}{3} \quad \frac{(a \quad b)x^{2}}{2} \quad abx]_{a}^{b} \mid
$$
\n
$$
| \frac{g''(\ )}{2} (\frac{b^{3}}{3} \quad \frac{(a \quad b)b^{2}}{2} \quad ab^{2} \quad \frac{a^{3}}{3} \quad \frac{(a \quad b)a^{2}}{2} \quad a^{2}b) \mid | \frac{g''(\ )}{12} | | (a \quad b)^{3} |
$$

The error term is very similar to the error term that we have for the Midpoint Rule, except that the error term is over 12 not 24. The error term for the Trapezium Rule when applied to I[f] is just the formula above with the usual change using Theorem 2.2 and so the error term is of the form

$$
e_m \quad \frac{(b \quad a)^3}{12} (k)^2 |f''( )|
$$

as defined in (2.16)

 The theorem above does make it seem that the error term for the Trapezium Rule is better than for the Midpoint rule but it is very misleading. The best way to look at the error is by the leading term order and the leading term order is  $k^2$ , which both of the methods have. The Midpoint Rule or the Trapezium rule can be a better approximation as the can be anywhere in [a,b] so the g''( ) term can be different for the Midpoint Rule and the Trapezium Rule. The Composite Trapezium Rule is constructed by the Trapezium Rule in the same way as the Composite Midpoint Rule is constructed from the Midpoint rule. We can have two ways that we can construct the Composite Trapezium Rule and in Definition 2.6 these two ways have been stated. **Definition 2.6** - The composite trapezium rule is defined as below when applied to a general function g(x) in [a,b].

$$
e_{\alpha} \quad \frac{h^2}{12} |g''(\ )|
$$
 [a,b] (2.22)

and so the error for the Composite Trapezium Rule when applied to I[f] is

$$
e_{\alpha} \quad \frac{h^2}{12} (k)^2 |f''( )|
$$
 (2.23)

## **Proof**

In the same way as we found the error for the trapezium rule is applied here. We

## wish to find  $|I - I_{ct}|$  where

```
N a jh
\mathbf{I}g(x)dx the exact solution I[f]
  j 1 a (j 1)h
   N a jh
         (cx d)dx the composite trapezium approximat ion to g(x), c, d are as defined earlier
I_{\text{ct}}j 1 a (j 1)h
0(Using6503Tm()Corepplie0946D<000e>TjM5310>Tj-1.299 nce0/TT4 1 Tse34n use005 T180 T5
```
**Example 2.2** - Once again a program has been created for the composite trapezium rule, which finds the exact and the approximate solutions and finds the absolute error. The table below shows the results that have been found. It uses the example 2.1 on with the trapezium rule and then as we can compare results. We should expect to get a similar results to the midpoint rule when k is small but when k is large we shouldn't see much of a difference.





The table below is once again how many intervals we need to get a 1% relative error and we can see that just like with the composite midpoint rule that as k doubles, we have to double N.

**Table 2.4** 

''		10	20	40	80	160	320	640
N	12 L	91	180	365	730	1460	2921	5842

## **The Simpson's Rule and the Composite Simpson's Rule**

The Simpson's rule is the third quadrature rule that we are going to look at in this project. This method is a better approximation than the last two methods because it uses more points to evaluate the solution. In fact the Simpson's rule uses the three

 From the diagram it is clear that the Simpson's rule is a lot better than the last two methods that we have looked at but the best way to look at this is to look at the error analysis

**Theorem 2.7** - The error of the Simpson's rule when applied to a general function  $g(x)$  in [a,b] is

$$
e_s \quad Sh^5 |g'''( )| \tag{2.27}
$$

and more so when the Simpson's rule is applied to I[f] we would expect an error of

$$
e_s \quad Sh^5(k)^4 \mid f'''( ) \mid \tag{2.28}
$$

#### **Proof**

The error term for the Simpson's rule is  $|e_s| = |I - I_s|$  where  $\mathbf b$  $g(x)dx$  The exact solution I[f]  $\mathbf{I}$  $\overline{a}$  $I_t$ <sup>b</sup>  $\frac{(f(a) - 4f(m) - f(b))}{3} dx$  The Simpson's rule approximation to g(x)

 The error term actually comes from finding definition 2.7. In finding definition 2.7 we also find the error term. We can start by using Taylor series on  $g(x)$  at the midpoint of  $[a,b]$ , which we will call  $x_1$ .

$$
g(x) = g(x_1) = g'(x_1)(x - x_1) = \frac{g''(x_1)}{2}(x - x_1)^2 = \frac{g^{(3)}(x_1)}{6}(x - x_1)^3 = \frac{g^{(4)}(x_1)}{24}(x - x_1)^4
$$

now as we are wishing to find the integral of  $g(x)$  we can integrate the above term and obtain

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 $\frac{11}{12}$  f<sup>'''</sup>

**Definition 2.8** - The composite Simpson's rule when applied to a general function g(x) is of the form

$$
\int_{a}^{b} g(x)dx \quad \frac{h}{3}[g(a) \quad g(b) \quad 2 \int_{n=1}^{N/2} g(x_{2n}) \quad 4 \int_{n=1}^{N/2} g(x_{2n-1})]
$$
 (2.31)

Where  $x_n = a+$  nh and h is the size of each interval with equal length.. The best way

to demonstrate this method is to look atdhise you can seeh i( each )]TJT\*c-0.0052 Tc-0.0102 Tw( inter(a

## **Proof**

If we follow through the proof of theorem 2.8, except we use a sum term and

k	N	Error e
1	$\overline{4}$	1.433047207E-04
	8	8.957255308E-06
	16	5.598312319E-07
	32	3.498946311E-08
	64	2.186841795E-09
	128	1.366777543E-10
10	$\overline{4}$	2.27130831E-01
	8	6.03838457E-03
	16	3.24500874E-04
	32	1.95783613E-05
	64	1.21307483E-06
	128	7.56535225E-08
100	4	1.70805212514
	8	1.70807325495
	16	1.70807460153
	32	5.57513418E-01
	64	8.85481914E-04
	128	4.29118644E-05

**Table 2.5** 

 If we compare the results that we have already got we can see that for a small k the Simpson's rule is by far the best at getting the smallest error. The results show that when k is equal to 1 then the error for the Simpson's rule is very good for only four intervals, but on the other hand when k is 100 the Simpson's rule is not very effective until we have 64 intervals, this is because of the (hk) ratio and we need h to be sufficiently small to get a good ratio.

**Table 2.6** 

.,	10	20	40	80	160	320	640
N	14	30	62	124	248	496	994

The table above is once again that as k doubles, then N has to also double to get a 1% relative error. This is the result because when looking at the error terms for all three methods used so far, the error term has the same ratio for h and k, therefore we get this result.

## **Gaussian Quadrature**

 The Gaussian quadrature method is the best quadrature method that we are going to look at; it gives 2N parameters to choose the weights and the nodes. For this method the nodes are not defined as  $(a+nh)$  or  $(a+(n-1/2)^*h)$  but as the zeros of the Legendre Polynomials. The problem with the Legendre polynomials is that there is not an analytical formula for them, just an iterative formula. This makes it difficult to compute the solution, fortunately there is an program in matlab that will find the zero's for a specific N, which is used to evaluate our solution. The reason that the Legendre polynomials are used is because th

are also optimal, this means that they have the greatest degrees of freedom, 2N, to fit a polynomial to the function  $g(x)$  and will fit a polynomial of degree 2N-1 to  $g(x)$  in each interval. Once again this has been proved before and won't be proved in this

Gaussian quadrature a few extra Ns have been added. These are added to show that Gaussian quadrature is good when N is small, when k is small.



#### **Table 2.7**

The table above shows that Gaussian quadrature is a lot better than any of the

Though not as clear as for the past few quadrature method the number of intervals N does roughly need to double when k doubles. This means that for all of the quadrature methods this property is observed. We can also notice that even though the values of N are small in the table below as k gets

# **3) Asymptotic Methods**

 In this section we will look at an asymptotic method called Integration by parts. It creates a finite term sum, found by integrating I[f] by parts N times.

**Definition 3.1** - The Integration by Parts approximation to I[f] in [a,b] is defined by



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 This method is a good method as it gets a better approximation as k gets large, which as we saw the quadrature methods do not do this. Unfortunately with the asymptotic method when k is small we would obtain a very poor result, and especially when k<1 then we would expect our error to increase when k gets smaller. The numerical results that we should expect to get would be that as k gets large our error should become less as N increases, but on the other hand a k gets small our error would grow. The next thing that needs to be solved is the way that we can program the integration by parts method. The problem we have here is that we have to be able to calculate derivatives and this can be very difficult to do. If we have a polynomial as our  $f(x)$ , the derivative is easy to calculate and it would be easy to produce a program for this. The same can be said for any sine, cosine and exponential functions. In short the derivatives that are in this method can be calculated by series that we determine.

**Example 3.1** - We can now look at our example 2.1 with the integration by parts method, and see how good at approximating it the asymptotic method is. Once again we are going to use the same k's and step sizes as before so that a good comparison can be made. An extra few N's in our table are added to show that for a very small number of intervals we get a very good result. We should find that for k=1 our answer is not going to give an answer anywhere near the exact one as can be seen by our error term that it is always going to be O(1) and will never home in on the answer. Other than for this we would not expect to get a good answer for k<1, but as k gets large it will be a very fast approximation.

 The second table is once again a table to show how many intervals are needed to get a 1% relative error. With this method once k get large a 1% error is obtained with just one approximation. When looking at the result we see that when k=1 we don't indeed have any good results, but when  $k = 10$  we get better results as N gets large and we soon get an error small enough to be equal to zero when using 18d.p. We should expect that as k gets larger we will get even better results, but if k is less than 1 we would get worse results. The Asymptotic Method is a very good method when we are looking at I[f], if k is large, but our results are clouded by one fact. In our example the derivative was calculated exactly because it never changed, but if we had to approximate the derivative then we would expect to get worse results depending on how good an approximation we use.

k	N	Error e
1	1	1.65101002708
	$\overline{2}$	1.65101002708
	$\overline{4}$	1.65101002708
	8	1.65101002708
	16	1.65101002708
	32	1.65101002708
	64	1.65101002708
	128	1.65101002708
10	1	3.580851422E-02
	$\overline{2}$	3.580851422E-03
	$\overline{4}$	3.580851422E-05
	8	3.580851406E-09
	16	6.206335383E-17
	32	The error is calculated
	64	as 0 after this
	128	
100	1	1.923703353E-04
	$\overline{2}$	1.923703353E-06
	4	1.923703321E-10
	8	Ω
	16	The error is calculated
	32	as 0 after this
	64	
	128	

**Table 3.1** 

# **4) Filon Type Methods**

$$
\int_{j=1}^{N} f(a \ (j \ 1/2)h) [\frac{e^{ikx}}{ik}]_{a \ (j=1)h}^{a \ jh} + \int_{j=1}^{N} f(a \ (j \ 1/2)h) (\frac{e^{ik(a \ jh)}}{ik} - \frac{e^{ik(a \ (j \ 1)h)}}{ik}) \ (4.2)
$$

For the result above we should expect to get a better result than we had for the normal midpoint rule. From what Arieh Iserles has written in his paper we should expect to get a very good result whether k is large or small, and it should be a combination of the asymptotic methods and the quadrature methods, which had problems when k was large or small.

**Theorem 4.1** - The error term of the Filon-Midpoint rule when applied to I[f] in [a,b] is

$$
e_{\text{fm}} \hspace{1.5cm} \text{Ch}^2 k \hspace{1.5cm} (4.3)
$$

**Proof** 

 The best way to look at the error is to try and find an approximation to it in terms of h and k. The best way to do this is to look at  $|e_m|=|I_{\text{fm}}|$  where

<sup>jh</sup> f(x)e<sup>ikx</sup>dx ,which is the exact solution  $\mathbf{L}$ j 1a  $(j-1)$ h  $\lim_{h \to 1} \frac{d\sinh}{d\sinh}$  f(a (j 1/2)h)e<sup>ikx</sup> dx which our Midpoint - Filon approximation  $\frac{1}{\text{fm}}$   $\frac{1}{\text{1a}}$   $\frac{1}{(1-\frac{1}{n})h}$ 

if we do subtract these then we obtain the expression

a [11/(2X)66512[203223343289.101]yex(T)T01/TA1288851310TD(f)Tj-42>Tj20.000D<0Tj6.997

 $1)$ h-j

 Looking at the first part of our integral as this is going to give the leading order term for the error. We can integrate this term by parts giving

$$
\frac{(h/2)e}{ik} \frac{(h/2)e}{ik} - \frac{1}{(ik)}(e \qquad e) ] |
$$

k	N	Error e
1	1	1.517042665E-01
	$\overline{2}$	3.833265725E-02
	4	9.606426383E-03
	8	2.403026322E-03
	16	6.008447781E-04
	32	1.502166984E-04
	64	3.755451848E-05
	128	9.388651111E-06
10	1	8.947847961E-02
	$\overline{2}$	1.449200059E-01
	$\overline{4}$	2.090240306E-02
	8	4.792239803E-03
	16	1.174630106E-03
	32	2.922420913E-04
	64	7.297279522E-05
	128	1.823772724E-05
100	1	1.670305201E-02
	$\overline{2}$	1.698160124E-02
	4	1.705566316E-02
	8	1.707445736E-02
	16	1.707917336E-02
	32	1.898678196E-04
	64	4.082912685E-05
	128	9.885567983E-06

**Table 4.1** 

 The tables above show us that in fact using the Filon methods for the midpoint rule we do get average results but not anywhere near as good as we have already got the only way we can see if this method is any good is by seeing what results we get when k is very large e.g. 10000. When we have k is 10000 and N being 100 for this method we get 6.8085749E-06 error, whereas for Gaussian integration we get 1.5120047e-02 error, and finally for the integration by parts method we get 5.5878500e-20. So it is clear here that the asymptotic method is by far the best method. There is also a very good property that this method has, when looking at table 4.2, (the table that shows how many intervals needed for a 1% relative error for specific ks) when k doubles the value of N does not double, which was th

**Definition 4.2**-The Filon- Trapezoidal approximation to I[f] is defined by

$$
\int_{a}^{b} f(x)e^{ikx} dx \int_{j=1}^{N} \frac{q(f(p)-pf(q))}{(q-p)} (\frac{e^{ikq}}{ik} - \frac{e^{ikp}}{ik})
$$
  

$$
\int_{j=1}^{N} (\frac{f(q)-f(p)}{q-p}) ((\frac{qe^{ikq}}{ik} - \frac{pe^{ikp}}{ik}) - (\frac{e^{ikq}}{(ik)^2} - \frac{e^{ikp}}{(ik)^2})) dx
$$

where  $q = a + ih$  and  $p = a + (i-1)h$ 

#### **Proof**

The trapezium rule works so that when we look in each interval we need to approximate  $f(x) = cx + d$ 

jh jh ikx ikx ikx dx dx cx dx

so to work this out we need to form simultaneous equations, which uses the information for the trapezium rule. These simultaneous equations are that if  $q=a + jh$ and  $p=a+(j-1)h$  then we get

$$
1) cp + d = f(p)
$$

$$
2) \text{ eq} + \text{ d} = f(q)
$$

if we work through the these equation we obtain that

c 
$$
\frac{f(q) - f(p)}{q - p}
$$
 and d  $\frac{q(f(p) - pf(q))}{q - p}$ 

and inserting this into our formula we obtain

$$
\begin{array}{ccccc}\n & & & & \text{N} & ^{a\text{ }jh} & & \\
1 & & f(x)e^{ikx}dx & & & \\
& & & & \text{A} & & \\
& & & & & \text{B} & \\
a & & & & & \text{B} & \\
& & & & & \text{C} & \\
& & & & & \text{D} & \\
& & & & & & \text{D} & \\
& & & & & & \text{D} & \\
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& & & & & & \text{D} & \\
& & & & & & \text{D} & \\
& &
$$

Now we have two integrals to work out, one which is just like the midpoint rule one which can be integrated directly, and one which we have to use integration by parts to find it, so there for we get the result



## **Proof**

To find the error we need to approximate  $|e_t|=|I_t|$  -  $I_t$ <sub>tt</sub> here

```
N a jh
            f(x)e^{ikx}dx The exact solution I[f]
\mathbf{I}j 1 a (j-1)h
    N a jh
```
 $(cx d)e^{ikx} dx$  The composite Filon - trapezoidal approximation to I[f], c, d are as defined earlier j 1 a  $(j-1)h$  $I_{tt}$ 

We can use Theorem 2.5 to find the term f(x)-cx-d to get

$$
|e_{\mathsf{f} t}|\mid \bigcap_{j=0}^{N}\frac{a\text{ }jh}{a\text{ } (j\text{-}1)h}\big(f(x)-cx-d\big)e^{ikx}dx\mid \big|\bigcap_{j=1}^{N}\frac{a\text{ }jh}{a\text{ } (j\text{-}1)h}\big(x-a\big)(x-b\big)e^{ikx}\mid
$$

In the same way that we saw this kind of term when looking at just the composite trapezium rule. So we need to integrate the above term, but firstly we need to gather like terms

$$
|e_{ft}| \mid \big| \sum_{j=1}^{N} \frac{f''( )}{2} \big[ \sum_{a (j=1)h}^{a jh} x^2 e^{ikx} dx - (a \quad b) \big] \sum_{a (j=1)h}^{a jh} xe^{ikx} dx \quad ab \quad e^{ikx} dx \big]
$$
 (4.11)

 Now we have three moments to work out and the best way to do this is to do one at a  $\overline{1}$ 

 the term above does look very complicated but many terms do cancel and so we are able to obtain an easier form

$$
|e_{ft}| \mid \mid \frac{N}{1} \frac{f''(1)}{2} [- \frac{qe^{ikq}}{(ik)^2} - \frac{pe^{ikp}}{(ik)^2} - \frac{pe^{ikq}}{(ik)^2} - \frac{qe^{ikp}}{(ik)^2} - \frac{2e^{ikq}}{(ik)^3} - \frac{2e^{ikp}}{(ik)^3}] |
$$

the form above is not in the form that we would like it to be in. We would wish to have the expression above in terms of h and k, but at the moment it is only in the form of k, p and q, so we wish to get the above expression in terms of (q - p) which is equal to h. To do this we can bring out  $e^{ikp}$  to the front to give

$$
|e| | \frac{f''(|e|)}{2(ik)} [(2-hik)e - (2-hik)]|
$$

k	N	Error e
1	1	1.373716326E-01
	$\overline{2}$	3.439271561E-02
	4	8.598958573E-03
	8	2.149751827E-03
	16	5.374381471E-04
	32	1.343595397E-04
	64	3.358988498E-05
	128	8.397471248E-06
10	1	8.561886880E-03
	$\overline{2}$	1.450586786E-02
	4	2.089185991E-03
	8	4.787185384E-04
	16	1.173221697E-04
	32	2.918810328E-05
	64	7.288197229E-06
	128	1.821498663E-06
100	1	1.668654671E-04
	$\overline{2}$	1.698124962E-04
	4	1.705581701E-04
	8	1.707451102E-04
	16	1.707918773E-04
	32	1.898675801E-06
	64	4.082873898E-07
	128	9.885451887E-08

**Table 4.3** 

**Table 4.4** 

''		$\overline{ }$ טו	ററ ∠∪	40	80	160	ാററ ∪∠ب	640
N	ັ	ററ ∠∪		$\sim$ ັ		ັ		

 The results above show that by using the trapezium rule as our quadrature base in the Filon type method we get by far better result than when using the midpoint. The Filon type method can also be used with the Simpson's rule as well, but as we know that the general pattern will give us a better approximation than the last two but not as good as with the Gaussian method. As has already been stated Arieh Iserles only works on the Gaussian in his paper and the first two that are in this paper will not give as good an error as Iserles puts in his paper but are a lot easier to calculate. The next section is what Iserles worked on in his paper and his method is followed through here.

#### **Filon - Gauss Method**

**Definition 4.3** - The form for Filon-Gauss method is below, where the  $I_n$  are Lagrangian interpolating polynomials, and  $x<sub>n</sub>$  are nodes determined by the zero's of Legendre polynomials.

$$
f(x)
$$
  $\tilde{f}(x)$   $\bigcup_{n=1}^{N} (x/h)f(x_nh)$ 

where

$$
I_i(x)
$$
  $\frac{N}{i}$   $\frac{x - x_i}{x_k - x_i}$   $\frac{1}{0}$   $\frac{x}{x}$   $\frac{x_k}{x_k}$   $j$  1,2,......N

This now can be put all together to get our approximation that we are looking for

$$
\int_{a}^{b} f(x)e^{ikx} dx \qquad \int_{a}^{b} \frac{N}{n(x/h)f(x_nh)e^{ikx}}
$$
\n
$$
\int_{a}^{b} \frac{1}{n(x/h)e^{ikx}dx} f(x_nh)
$$

 Now the part of the method that is the most important is the weights, as we need to be able to work these out exactly. The way we do this is that we see that there is a pattern of the integrals that we need to be able to solve for, they are of the form

$$
b_n \int_{a}^{b} x^n e^{ikx} dx
$$

These integrals are called moments as we have seen then before, the way that these are worked out is by using asymptotic methods to get the leading term order of the integrals. If we integrate our form by parts then we obtain the formula

$$
b_n \xrightarrow[a]{b} x^n e^{ikx} dx \frac{[b^n e^{ikb} - a^n e^{ika}]}{ik} - \frac{1}{ik} \int_{a}^{b} nx^{n-1} e^{ikx} dx
$$

**Theorem 4.3** - The error term for the Filon-Gauss method when applied to I[f] is when

$$
(hk) << 1 \qquad \qquad \text{error} = \ O(h^{N+1})
$$

$$
(hk)=O(1) \terror = O(h^{N+1})
$$

(hk)>>1 \terror = 
$$
O(\frac{h^{N-1}}{k})
$$
 If endpoints a and b are not

included in the nodes x

R[f] 
$$
\frac{1}{(ik)^{N-1}}\int_{a}^{b} (\frac{f(1)}{g'(1)})^{(N)} e^{ikx} dx
$$
 a b

 Now we have the problem when there is a stationary point in our interval i.e. if  $g'(x)=0$  in [a,b]. What we do is separate the interval in to pieces, we will have the small area around the stationary point taken out and use the integration by parts method on the bits without the stationary point, and then work on the bits that do. **Theorem 5.1** - If we have a stationary point at say the point x=a and no where else in our interval then the leading term is of order  $O((1/k^{1/2}))$ 

#### **Proof**

The procedure follows the following steps

$$
\int_{a}^{b} f(x)e^{ikg(x)}dx \qquad \int_{a}^{a} f(x)e^{ikg(x)}dx \qquad \int_{a}^{b} f(x)e^{ikg(x)}
$$

this is the split interval, we can see that the area around the stationary point is now isolated by a small parameter and the second interval can be solved as before, using the integration by parts method

The second interval is solved in the following way

$$
\int_{a}^{b} f(x)e^{ikx} dx = \int_{n-1}^{N} \frac{1}{(ik)^{n}} [((\frac{f(x)}{g'(x)})^{(n-1)}e^{ikg(x)})]_{a}^{b} = \frac{1}{(ik)^{N-1}}_{a}^{b} = (\frac{f(x)}{g'(x)})^{(N)}e^{ikx} dx
$$

 It is clear that there is no irregular points in this integral (unlike if a was included) and we obtain an inverse of k leading order term with a remainder term like before.

 The second integral that we need to work out is different, what we need to do is to remove the irregular point, we do this by using Taylor series on  $g(x)$  and  $f(x)$ .

a 
$$
(ik\frac{x-a}{2}]g''(a)
$$

## **6) Filon Type Methods**

## **Filon-Trapezoidal Method**

When using Filon type methods on our integral J[f] we have to be careful with the  $g(x)$  term in the exponential. If  $g'(x) = 0$  in [a,b] then an easy approximation can be found by using a combination of the Filon method and the Integration by parts method. On the other hand if  $g'(x)=0$  at least once in [a,b] then we would have to use method of stationary phase instead. The Filon-Trapezoidal method for J[f] follows the same line as we did before up to where we need to integrate terms.

**Definition 6.1** - The Filon-Trapezoidal approximation to J[f] on [a,b] is defined as

 $\mathsf b$ N a1 jh  $j<sub>1</sub>$  $\mathsf{a}$ 

Case 2 - Integrals 
$$
\int_{a}^{b} xe^{ikg(x)}dx
$$
 and  $e^{ikg(x)}dx$  are not known, but g'(x) 0 in [a,b].

In this case we need to use the integration by parts method on these integrals. We can

use Definition 5.1. For the integral 
$$
e^{ikg(x)}dx
$$
 the method is  
\n $e^{ikg(x)}dx$   $\int_{a}^{b} \frac{g'(x)}{g'(x)}e^{ikg(x)}dx$   $\left[\frac{e^{ikg(x)}}{ikg'(x)}\right]_{a}^{b} - \frac{1}{ik} \int_{a}^{b} \left(\frac{1}{g'(x)}\right)^{i}e^{ikg(x)}$ 

we can follow this through for as many terms as we wish to get a good approximation. We get a sum term of the form

$$
\int_{a}^{b} e^{ikg(x)} dx \int_{n-1}^{N} \frac{1}{(ik)^{n}} \left( \left( \frac{1}{g'(x)} \right)^{(n)} e^{ikg(x)} \right) \int_{a}^{b} \frac{1}{(ik)^{N}} e^{i} \left( \frac{1}{g'(x)} \right)^{(N)} e^{ikg(x)} dx
$$

 $\mathsf b$ for the integral  $xe^{ikg(x)}dx$  we need to follow through a similar approximation  $\mathsf{a}$ 

$$
\int_{a}^{b} xe^{ikg(x)} dx \int_{a}^{b} \frac{x'g(x)}{g'(x)} e^{ikg(x)} dx \left[ \frac{xe^{ikg(x)}}{ikg'(x)} \right]_{a}^{b} - \frac{1}{ik} \int_{a}^{b} \frac{x}{g'(x)} e^{ikg(x)} dx
$$

once again we get a sum term depending on how many approximations we wish to have and this is of the form

$$
e^{ikg(x)}dx
$$
  $\int_{n_1}^{N} \left[\frac{1}{(ik)^n} \left(\left(\frac{x}{g'(x)}\right)^{(n-1)} e^{ikg(x)}\right)\right]_a^b \frac{1}{(ik)^N} e^{i(kx)} dx$ 

Now we can put all of these terms together and substitute them into (6.1)

$$
\begin{array}{ccccc} \ _{N} & N & N \\ & \ _{j\ 1\ n\ 1} & \frac{c}{(ik)^{n}}((\frac{x}{g'(x)})^{(n\text{-}1)}e^{ikg(x)})]_a^b & \frac{d}{(ik)^{n}} & ((\frac{1}{g'(x)})^{(n\text{-}1)}e^{ikg(x)}) \\ & \ _{a} & \end{array}
$$

the above term does look complicated but it is just a combination of the integration by

**Case 3 - Integrals** 
$$
x e^{ikg(x)} dx
$$
 and 
$$
e^{ikg(x)} dx
$$
 are not known, and  $g'(x) = 0$  in [a,b].

In this case the method of stationary phase must be used to approximate the moments. In this case we would have to take into account all of the possible outcomes of the method of stationary phase method (theorems 5.1, 5.2 ,5.3 and 5.4). Each different case would affect the leading term behavior of this method and in the same way that definition 6.1 was created by the integration by parts method, a procedure would be created in the same mould. Unfortunately the method of stationary phase was not defined as the integration by parts method was and much work would need to be put into this method, so only a brief overview can be added here. It is clear that depending on which case we have from section 5) will have a big affect on the final answer and the leading term behavior will affect the final approximation.

## **Conclusion**

 In this project many different types of methods have been looked at on the two integrals that we are interested in. For the first integral I[f] we have looked at 3 different types of methods, Quadrature methods, Asymptotic methods and Filon type methods. Out of these three methods the Asymptotic method was by far the best when k is large, giving brilliant results, but when k is small the method gives very poor results. The quadrature methods gives very poor results for k large, and the best of these methods, the Gaussian quadrature method gets very good results for small k but as k gets large the error grows so that we would have to decrease h so much that it becomes very expensive to calculate the approximation. The third and final type of method we looked at was the Filon type methods. These methods do decrease the

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 The second integral that we looked at J[f] we looked at two different types of methods, but not in any great detail. We did find that when we get a  $g(x)$  instead of  $x$ we get very difficult analysis, with a lot of integration that needs to be approximated by asymptotic methods. Unfortunately this was not completed and only a brief overview was seen with this method, but as with the first integral I[f], the Filontrapezoidal method was a very good compared with the other method that were looked at, we should expect to get similar results on J[f].

## **7) Bibliography and Literature Review**

**1) On the Numerical Quadrature of Highly-Oscillating Integral I: Fourier transforms - Arieh Iserles** - **IMA Journal of Numerical Analysis 2004 p365-391** This is a paper that was published by the Cambridge Professor Arieh Iserles in 2003. It deals with the integral I[f] when  $k = 0$ , so that the paper is based on Fourier transforms. The first part of the paper deals with Gaussian Quadrature, in this section it is stated that Gaussian quadrature is the best possible Quadrature method available and follows the procedure that the Gaussian Quadrature method is too expensive to find a suitable approximation to I[f]. Though this is true, Gaussian quadrature does give good results for an averagely big k but no where near as good as the filon-Gauss method described here in this paper. The next part of the Paper deals with the Filon-Gauss method, it gives the error estimates that are given in Theorem 4.3, these error estimates are very good and if they are accurate then this is by far the best result that has been seen for oscillatory integrals. The problem is that when in the paper he works through his error estimate proof it is very complicated and is very hard to follow, this means that not only is the analysis hard, nut also the procedure of the method is hard too. The last part of the paper deals with other types of methods that can be used also. These methods are Zamfirescu's Method, Levin's Method and lastly a revisit of lie group Methods.

**2) Numerical Analysis - Richard L. Burden and J. Douglas Faires- 6th edition 1997 Brooks/Cole publishing company** 

This is a book, and the pages that I dealt with were mainly were p188-205 and p222- 228.

The first set of pages deal with the first three quadrature methods in this paper, the Midpoint rule, Trapezium rule and the Simpson's rule. The book firstly looks at all of the General methods and errors in chapter 2.2, and in 2.3 the book goes on to composite rules. All of the error estimates that are included in this paper are derived from these chapters and also other parts of the book have been used to prove these error estimates. The second set of pages deal with Gaussian quadrature, the pages state the Gaussian quadrature method and how it works but doesn't work through any error estimates.

**3) Advanced Mathematical Methods for scientists and engineers - C.M. Bender & S.A. Orszag 1978 - p252-261 and p276-280** 

 This source is where the asymptotic type methods in this project were found. The first set of pages deal with the integration by parts method found in chapter 3. the second set of pages deal with the Method of Stationary Phase found in chapter 5.

**4) On the numerical quadrature of highly-oscillating integrals II: Irregular Oscillators - Arieh Iserles – IMA journal of Numerical Analysis 25 (2005) p25-44.** 

 This is the second paper that Arieh Iserles follows through the numerical quadrature of highly oscillating integrals. The first part of the paper is a recap of the first paper 1) giving the error term only in terms of and not of h as well. This is good though as it shows that if h is large then the method does still get a good result if k is also large. The paper then moves on to the second integral J[f] with respect to Gaussian quadrature, (which is not dealt with in this project, but the basis of the Filon-Trapezoidal method is to see if it has easier analysis.) He beginsl J[f Tc 05 Tw [(oT0005(i)-1.92.4524.7603 0 o0001 Tc

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follow through and most of it could be skipped through, but Iserles shows here that he does look through all types of scenarios.

**5) Efficient quadrature of highly oscillatory integrals using derivatives - Arieh Iserles – Proceedings of the Royal Society A (2005)**

 This is the third paper Iserles published on highly oscillating integrals. This paper deals with asymptotic method applied with Filon's ideas. The paper begins by looking at J[f] when there are no stationary points and working through the integration by parts method that is in section 5) of this thesis. The notation of the paper is very good in this section defining

$$
0 = f(x)
$$
  

$$
k + \frac{d}{dx} \frac{k[f](x)}{g'(x)}
$$

This notation goes very well with the method and his makes the whole method very easy to understand. He then uses the integration by parts method with Filon