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The elliptic sin-Gordon equation in a half plane

by

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of a nonlinear integrable PDE, equation (1.1) is a time-independent PDE of elliptic type, and therefore it differs from most other one-dimensional nonlinear integrable models, that describe a temporal evolution process.

The inverse scattering method has been used to analyse this equation in $\mathbb{R}^2$; namely, the problem with prescribed periodic behaviour at infinity was considered in [5], while special solutions for the problem posed in the whole of $\mathbb{R}^2$ were found in the 80's, see the references in [5]. However, the classical inverse scattering transform cannot be used in general to derive a solution representation without adapting it to allow for the treatment of boundary conditions.

Such an extension of the classical inverse scattering transform has recently been proposed and applied to solve a variety of boundary value problems for integrable evolution PDE, see the monograph [6]. In this paper, we use this extension to analyse boundary value problems for (1.1) posed in the half plane $\{(x, y) : x \in \mathbb{R}, y > 0\}$. Such problems were also considered in [10] under the assumption that the boundary data satisfy a certain nonlinear equation, deduced heuristically by analogy with the linearized case. We show here that this nonlinear equation is obtained as a consequence of the so-called *global relation*. The global relation is derived rigorously in our approach, and it is shown to imply a stronger constraint on the boundary data than the one imposed in [10]. By requiring its validity, we characterize all pairs of functions that can occur as boundary values of decaying (mod 2π) solutions of equation (1.1) in the half plane.

We also briefly consider the problem of solving a generic well-posed boundary value problem, when one boundary condition is prescribed and a second one must be determined. An example of such problem arises when the Dirichlet datum $q(x, 0)$ is given while the Neumann datum $q_y(x, 0)$ is unknown and must be obtained as part of the solution. The characterization of the unknown boundary value relies on the analysis of the global relation. The experience

The matrix-valued function Q_0 satisfies the symmetry properties

$$Q_0(x;\ )_{22} = Q_0(x;\ )_{11}; \qquad Q_0(x;\ )_{12} = Q_0(x;\ )_{21} \tag{2.3}$$

These properties guarantee that for $\ \in \mathbb{R}$ the matrix $m(x;\ )$ is well defined as the unique solution of the linear integral equation

$$m(x;\ ) = I \quad \int_x^{\infty} e^{-\frac{\omega_2(\ )}{2}(x-\ )\,\sigma_3}\,[Q_0(\ ;\ )m(\ ;\ )]\,d\ : \tag{2.4}$$

Indeed, the existence and uniqueness of the solution of this linear integral equation, under the given assumptions and given the symmetry properties of Q_0, can be established by adapting the proofs of the more general results in [3, 4]. See also the section on the sine-Gordon equation in [7].

Definition 2.1 *Let g_0, $g_1 : \mathbb{R} \to \mathbb{R}$ satisfy the two properties* (p1) *and* (p2)*, and let $m(x;\ )$ be the matrix defined as the unique solution of the linear integral equation (2.4), with Q_0 given by (2.2). Define the matrix-valued function $R(\ ) : \mathbb{R} \to \mathbb{C}$ by*

$$R(\ ) = \lim_{x\to-\infty} \Big[e^{\frac{\omega_2(\ )}{2}x}$$

Proposition 2.1 *Let q be a solution of equation (1.1) in the half plane $\{y \geq 0\}$, such that $q \in C^2(\mathbb{R} \times \mathbb{R}^+)$ and satisfies $q + 2\pi m$, $q_y \to 0$ when $|y| + |x| \to \infty$ ($m \in \mathbb{Z}$). Let*

$$g_0(x) = q(x,0); \qquad g_1(x) = q_y(x,0).$$

Then the functions $\{g_0(x), g_1(x)\}$ form an admissible set.

Proposition 2.2 *Let $\{g_0(x), g_1(x)\}$ be an admissible set, and let $b(\lambda)$ be its associated spectral function.*
Let $M(x,y,\lambda)$ be the unique solution of the following Riemann-Hilbert problem :

$$M_-(x,y,\lambda) = M_+(x,y,\lambda)J(x,y,\lambda); \qquad \lambda \in \mathbb{R}; \qquad \det(M_\pm) = 1; \tag{2.8}$$

where $M_\pm$ are analytic functions of λ in $\mathbb{C}^\pm$ respectively, and

$$J(x,y,\lambda) = \begin{pmatrix} 1 & b(\lambda)e^{-\theta(x,y,\lambda)} \\ b(\lambda)e^{\theta(x,y,\lambda)} & 1 \end{pmatrix}; \qquad \theta(x,y,\lambda) = \omega_1(\lambda)y + \omega_2(\lambda)x. \tag{2.9}$$

This Riemann-Hilbert problem is uniquely solvable, and the function $q(x,y)$ defined by

$$q_x - iq_y = -(\tilde{M})_{12}; \qquad \cos q(x,y) = 1 - 4i(\frac{\partial}{\partial x}\tilde{M}_{22}) + 2(\tilde{M}_{12})^2; \tag{2.10}$$

where

$$\tilde{M} = \lim_{\lambda\to\infty}(\lambda(M - I)); \qquad I = diag(1,1);$$

solves the following boundary value problem for the elliptic sine-Gordon equation:

$$q_{xx} + q_{yy} = \sin q; \qquad x \in \mathbb{R},\ y \geq 0; \tag{2.11}$$

$$q(x,0) = g_0(x); \quad q_y(x,0) = g_1(x); \qquad x \in \mathbb{R}. \tag{2.12}$$

The proof of these two propositions is given in section 4. In the next section we illustrate the main steps of our Riemann-Hilbert approach in the simpler linear case.

3 The modified Helmholtz equation in the half plane

We consider the linear version of the elliptic sine-Gordon equation (1.1), an important equation in its own right known as the *modified Helmholtz* equation. This equation, in the context of the method we use in this paper, is studied in [6], where full details can be found. For concreteness, we briefly analyse a concrete, Dirichlet boundary value problem for this equation in the half plane:

$$q_{xx} + q_{yy} = q; \qquad x \in \mathbb{R},\ y > 0; \tag{3.1}$$

$$q(x,0) = g_0(x); \qquad x \in \mathbb{R}; \tag{3.2}$$

where the function $g_0(x)$ is assumed to have appropriate smoothness and decay at infinity (for example, to satisfy the properties (p1) and (p2)).

In this expression, the function $\tilde{g}_0(\ \ )$ can be computed from the given Dirichlet boundary condition, but the function $\tilde{g}_1(\ \ )$ is unknown. The remaining problem is the determination of the transform $\tilde{g}_1(\ \ )$ of the unknown boundary value $q_y(x; 0)$:

To this end, we consider the di erential form $W(x; y;\ \ )$, which is bounded in for all $(x; y)$ in the half plane when $\ \ < 0$. This form is exact, hence it is also closed in the half plane $y > 0$, and therefore

As in the linear case, the Lax pair is equivalent to the condition that the (matrix-valued) di erential form , given by

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Since $M_2(x;$

The Riemann-Hilbert problem

We continue our analysis, under the assumption that q is a solution of (1.1) with appropriate smoothness and decay. Our aim is to give a representation of q in terms of its boundary values. To this end, in this section we use the outcome of the spectral analysis to determine explicitly and to solve a Riemann-Hilbert problem. We then show that q can be expressed in terms of the solution of this problem.

The condition (4.10) and the asymptotic condition (4.6) determine uniquely a matrix Riemann-Hilbert problem on $\mathbb{R}$. Indeed, defining

$$\begin{aligned} M_+(x;y;\lambda) &= (M_2^+;M_1^+) \quad \lambda \in \mathbb{C}^+; \\ M_-(x;y;\lambda) &= (M_1^-;M_2^-) \quad \lambda \in \mathbb{C}^-: \end{aligned}$$

we find by rearranging (4.10) that

$$M_-(x;y;\lambda) = M_+(x;y;\lambda)J(x;y;\lambda); \qquad \det(M_\pm) = 1;$$

where the jump matrix $J(x;y;\lambda)$ is given by (2.9). Since the global relation implies that $b(\lambda) = 0$ for $\lambda \in \mathbb{R}^+$, the matrix J is either upper or lower triangular for each $\lambda \in \mathbb{R}$.

Rewriting the jump condition, we obtain

$$M_+ - M_- = M_+ - M_+J = M_+(I - J) \Rightarrow M_+ - M_- = M_+\tilde{J} \tag{4.16}$$

where $\tilde{J} = I - J$.

The solution of this Riemann-Hilbert problem is now given by a standard Cauchy-type formula, see e.g. [1]. For example, the second column of the solution $M(x;y;\lambda)$ of this RH problem is given by

$$\begin{pmatrix} M_{12}(x;y;\lambda) \\ M_{22}(x;y;\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2\pi i}\int_0^\infty \begin{pmatrix} (M_+\tilde{J})_{12}(x;y;\lambda') \\ (M_+\tilde{J})_{22}(x;y;\lambda') \end{pmatrix} \frac{d\lambda'}{\lambda' - \lambda} = \tag{4.17}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2\pi i}\int_0^\infty \begin{pmatrix} (M_+)_{11}(x;y;\lambda')b(\lambda')e^{\theta(x;y;\lambda')} + (M_+)_{12}(x;y;\lambda') \\ (M_+)_{21}(x;y;\lambda')b(\lambda')e^{\theta(x;y;\lambda')} + (M_+)_{22}(x;y;\lambda') \end{pmatrix} \frac{d\lambda'}{\lambda' - \lambda}:$$

and similarly for the first column (with integral along the negative real axis).

The characterization of $q(x;y)$

To characterize q in terms of the solution of the Riemann-Hilbert problem, we consider the asymptotic estimate (4.6) and let

$$\tilde{M} = \lim_{\lambda\to\infty}(\lambda(M - I)):$$

Substituting (4.6) into the ODE (4.1), we find that the coefficient of λ^0 yields

$$\frac{i}{4}[\sigma_3;\tilde{M}] = iQ$$

Proof of Proposition 2.2

Up to now we have assumed that $q(x; y)$ was a suitable solution of equation (1.1) posed in the half plane, and have shown that this function can be represented through the solution of the associated Riemann-Hilbert problem (4.16). The data of this Riemann-Hilbert problem are constructed in terms of the boundary values of q at $y = 0$.

We now start from a pair of such data, assuming that they form an admissible set. Using the spectral funcion $b(\ )$ associated with this set, we can determine a Riemann-Hilbert problem as in the statement of the proposition, with $q(x; 0)$ and $q_y(x;$

Let

$$U(x,\lambda) = Q(x,0,\lambda) - \frac{w_2(\lambda)}{2}\sigma_3 = \frac{iq_x + q_y}{4}\sigma_1 - \frac{i}{4}(\sin q)\sigma_2 + \frac{i}{4}\left(\lambda - \frac{1}{\lambda}\cos q\right)\sigma_3,$$

where q, q_x q_y are evaluated at $y = 0$. It is immediate to verify that $\det(U)$ is a function of λ only through $w_2(\lambda)$. Since $w_2(-\frac{1}{\lambda}) = w_2(\lambda)$, we have $\det(U(-\frac{1}{\lambda})) = \det(U(\lambda))$. Hence it is natural to seek a matrix $T = T(\lambda)$, a function of λ *only*, satisfying

$$U(x,-\frac{1}{\lambda})T(\lambda) = T(\lambda)U(x,\lambda), \quad x \in \mathbb{R}. \tag{5.2}$$

If such a matrix $T(\lambda)$ exists, then it must also satisfy

$$m(x,-\frac{1}{\lambda})T = Tm(x,\lambda), \quad \forall x \in \mathbb{R}. \tag{5.3}$$

Since

$$b(\lambda) = \lim_{x\to-\infty}\left(e^{\frac{w_2(\lambda)}{2}x\sigma_3}m(x,\lambda)\right)_{21}$$

the relation (5.3), in the limit as $x \to -\infty$, implies a relation between $b(\lambda)$ and $b(-\frac{1}{\lambda})$. This relation, and the global relation, could then be used to express $b(\lambda)$ is terms of only one boundary condition.

Thus we have reduced the problem to finding a matrix T that satisfies the condition (5.2). Imposing this condition, we find (up to multiples)

$$T = T(x,\lambda) = \begin{pmatrix} 1 & \frac{1+\lambda^2}{1-\lambda^2}\frac{1-\cos q(x,0)}{i\sin q(x,0)} \\ \frac{1+\lambda^2}{1-\lambda^2}\frac{1-\cos q(x,0)}{i\sin q(x,0)} & 1 \end{pmatrix} \tag{5.4}$$

The matrix-valued function T is independent of x only if $q(x,0) = constant$. However, the decay requirement on $q(x,0)$ (mod 2π) as x tends to infinity then implies that $q(x,0) = 2\pi m$, for some $m \in \mathbb{Z}$, and the unique solution of the problem is the zero solution (mod 2π). Hence there are no nontrivial linearisable conditions associated with the Lax pair (4.1)-(4.2).

Remark 5.1 Linearisable conditions are associated with a specific choice of Lax pair, and results of [10], finding a representation of the solution of (1.1) under the assumption that all boundary values are prescribed in such a way that the global relation is satisfied.

Two properties of this problem are worthy of mention, as they appear to be specific to this model and differ from the analogous properties of the usual sine-Gordon equation, which describes a time evolution process.

Firstly, our results imply that the "nonreflecting"case $b(\ ) = 0$ considered in [10] is not compatible with any choice of admissible boundary conditions. It is natural to consider the "nonreflecting" case in the context of evolution equations, when it is then possible to compute explicitly pure soliton solutions. The formal computation can be performed for the abstract Riemann-Hilbert problem defined in this paper, as done in [10], but it does not correspond to a well-posed boundary value problem of the kind examined here (indeed, the explicit soliton solution obtained in [10] does not decay in all directions). Assuming that the solution decays (mod 2), we have shown that, for admissible boundary values, $a(\ ) = 1$, 2 R, and if in addition $b(\ ) = 0$ then the Riemann-Hilbert problem is trivial ($J = I$) and $M = I$, implying that $q(x;y) = 0$ (mod 2). This is a significant property, to our knowledge previously undetected, specific to the present problem.

Secondly, it appears that it is not possible to prescribe boundary conditions for which the solution representation is explicit. Such conditions exist for other integrable evolution PDEs (the NLS and KdV equations, as well as sine-Gordon) and are called *linearisable*. In the present case, the conjugating matrices that characterize linearisable boundary conditions are trivial. We showed this here for the construction associated with the Lax pair (4.1)-(4.2). However, it can be shown that the same holds true when basing the construction of linearisable boundary conditions on the alternative Lax pair used in [8]. Hence it appears that no linearisable boundary conditions can be prescribed for the elliptic sine-Gordon equation posed on a half plane.

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References

[1] M.J Ablowitz and A.S. Fokas, *Introduction and applications of complex variables*, Cambridge University Press, 2nd ed. 2003.

[2] M.J Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, Method for solving the sine-Gordon equation, *Phys. Rev. Lett.* (1973) 30, 1262-1264

[3] R Beals and R.R. Coifman, Scattering and inverse scattering for first order systems, *Comm. Pure Appl. Math.* (1984) **37**, 39-90.

[4] R Beals and R.R. Coifman, Scattering and inverse scattering for rst order systems: II, *Inv. Prob.* (1987) **3**, 577-593.

[5] A.B. Borisov and V.V. Kiseliev, Inverse problems for an elliptic sine-Gordon equation with an asymptotic behaviour of the cnoidal type, Inv. Prob. (1989) 5, 959-982.

[6] A.S. Fokas, A uni ed approach to boundary value problems, SIAM monographs (2008).

[7] A.S. Fokas, A uni ed transform method for solving linear and certain nonlinear PDE's, Proc. Royal Soc. Series A (1997) 453, 1411{1443.

[8] A.S. Fokas, Linearizable initial boundary value problems for the sine-Gordon equation on the half line, Nonlin. (2004) **17**, 1521-1534.

[9] A.S. Fokas and B. Pelloni, *Boundary value problems for the elliptic sine-Gordon equation in a semi-strip*, preprint (2009)

[10] E.S. Gutshabash and V.D.Lipovskii, Boundary value problem for the two-dimensional elliptic sine-Gordon equation and its applications to the theory of the stationary Josephson e ect, J. Math. Sciences (1994) 68, 197-201.

[11] V.D. Lipovskii and S.S. Nikulichev, Vestn. LGU Ser. Fiz. Khim. (1988) 4, 61-34

[12] B. Pelloni. The asymptotic behaviour of the solution of boundary value problems for the sine-Gordon equation on a nite interval, J. Nonlin. Math. Phys. (2005) 12(4), 518-529.