
Linear and Quadratic Finite Elements for a Moving Mesh Method

By

Muhammad Akram

Supervised By

Professor Mike Baines

A dissertation submitted to the Department of Mathematics, the
University of RAikuhammaRAiku(k)dr5rlhe

Declaration

I hereby declare that this dissertation has not been accepted in substance for any degree and not being concurrently submitted for any other degree.

I confirm that this is my own work and the use of all materials from other sources has been properly and fully acknowledged.

Muhammad Akram

Candidate's Signature

Supervisor's Signature

Abstract

In this Dissertation linear and quadratic finite elements are used to produce numerical approximation to the solutions of first order differential equations which arise in a moving mesh finite element method. The behaviour of the moving mesh velocity is investigated in detail and is compared these results with the existing exact solutions to investigate the effect of the moving boundaries and provided the error analysis in both linear and quadratic cases.

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Chapter 1

Introduction

This dissertation concerns the finite element solution of first order differential equations. It is well known that the standard finite element method is not well suited to $\frac{dy}{dx} = f(x)$ in $(0, 1)$ with $y(0) = 1$, due to the insufficient boundary conditions and difficulty of inverting the element matrix, leading to various kinds of regularisations in the literature.

There are many applications in which first order equations arise, notably in steady state fluid mechanics. The motivation in this dissertation, however comes from a moving mesh method for time-dependent Partial Differential Equations (PDEs).

The strategy is to replace the first order differential equation by a second order one with an artificial boundary condition, giving $(\frac{dy}{dx} = \frac{d^2u}{dx^2} = f(x)$ with $\frac{du}{dx}(0) = 1$ and $u(1) = 0$) a problem which is well suited to the finite element approach. It then remains to recover the solution of the first order equation from the finite element solution obtained. In moving mesh applications this has to be a continuous function.

The moving mesh work is new in this field and therefore, there is a limited amount of information available in existing literature. Due to the nature of this dissertation, the majority of the preliminary work is based on programming.

1.1 Moving mesh velocity equation

A moving mesh approach to solving the Partial Differential Equation (PDE)

$$\frac{\partial p}{\partial t} = L_x p$$

where L_x is a partial operator, is to use conservation of the integral of p to move the mesh,

$$\frac{d}{dt} \int_0^{\hat{x}(t)} p dx = 0$$

By Leibnitz' Integral Rule [

for the vector of velocities Y , where B is a matrix with entries

$$B_{ij} = \int_a^b \rho \frac{d\varphi_i}{dx} \varphi_j dx$$

This is an unsymmetric matrix, similar to an anti-symmetric matrix, and difficult to invert.

1.3 Alternative approach

An alternative approach is to write $y = \frac{du}{dx}$ where u is a velocity potential, giving the second order equation

$$-\frac{d}{dx}\left(\rho \frac{du}{dx}\right) = L_x p$$

with weak form

$$-\int_a^b \omega \frac{d}{dx} \left(\rho \frac{du}{dx} \right) = \int_a^b \omega L_x p dx$$

giving, after integration by parts

$$-\varphi_i \rho \frac{du}{dx} \Big|_a^b + \rho \frac{d\varphi_i}{dx} \left(\sum_{ij} \right)$$

1.4 Finite elements for a first order Differential Equations

Consider solving the first order differential equation problem

$$\frac{dy}{dx} = f(x), \quad y(0) = 1 \quad (1.1)$$

in $(0,1)$ by the Finite Element method. The weak form is

$$\int_0^1 \omega_i \frac{dy}{dx} = f(x)$$

Replace $\omega_i = \varphi_i(x)$ by piecewise linear or quadratic basis functions and expand

$$y \approx Y = \sum Y_j \varphi_j$$

which gives us

$$\sum_{j=0}^1 Y_j \quad Y_j \quad \sum$$

B is badly conditioned, anti-symmetric and difficult to invert.

In the approach used in this dissertation, put

$$y = \frac{du}{dx}$$

and instead of solving (1.1), solve the second order equation

$$\frac{d^2 u}{dx^2} = f(x) \quad (1.3)$$

where $\frac{du}{dx} = 1$ at $x = 0$ and we impose (arbitrarily) the artificial boundary condition $u = 0$ at $x = 1$. From the weak form of (1.3), we have

$$\omega_i \frac{du}{dx} \Big|_0^1 - \int_0^1 \frac{d\omega_i}{dx} \frac{du}{dx} dx = \int_0^1 \omega_i f(x) dx$$

Replacing ω_i by finite element basis functions $\phi_i(x)$ and approximating u by

$$U = \sum_{j=0}^{N-1}$$

1.5 Outline of the Dissertation

Chapter Two, is based on theory of finite elements for second order differential equations. This chapter investigate the method of Linear Finite elements and deficiencies in this method for our purpose and provides an alternative Quadratic elements method to find the numerical solution.

Chapter Three, provides the Linear and Quadratic approaches to solve the first order differential equations as well as the Sturm-Liouville type differential equations. In this

Chapter 2

Finite Elements for Second order Equations

There are many ways to solve Partial Differential Equations (PDEs) numerically with advantages and disadvantages. The Finite Element Method (FEM) is a good choice for solving PDEs over complex domains, when a domain changes (as during a solid state reaction with a moving boundary), when the desired precision varies over the entire domain, or when the solution lacks smoothness. For instance, in simulations of the weather patterns on Earth, it is more important to have accurate predictions over land than over the open sea, a demand that is achievable using the finite element method.

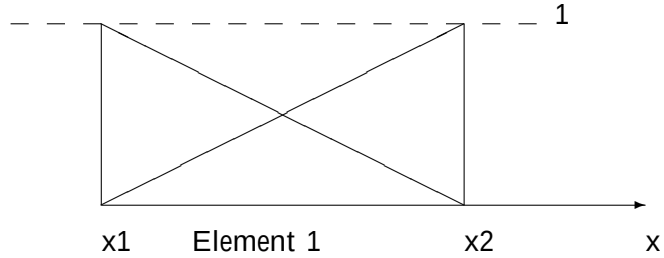


Figure 2.1: Linear finite elements for one element

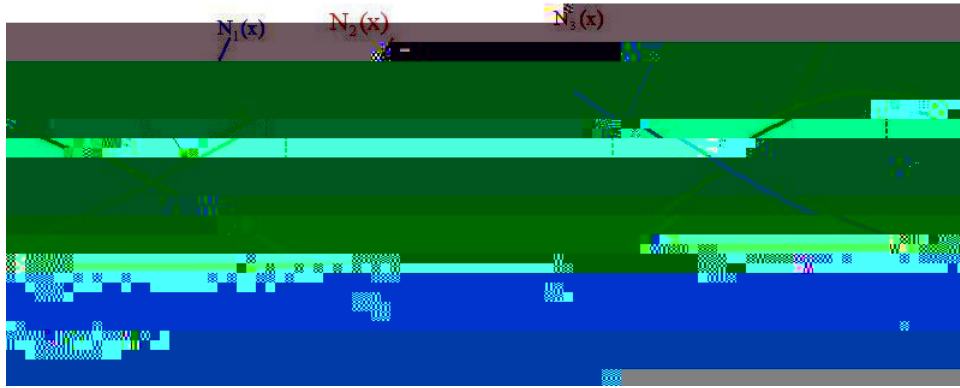


Figure 2.2: Quadratic finite elements for one element

C^1 and square integrable in (a,b) and integrate it, giving

$$-\int_a^b \omega_i(x) \frac{d^2 u}{dx^2} dx = \int_a^b \omega_i(x) f(x) dx \quad (2.2)$$

Now integrate left hand side of (2.2) by parts, giving

$$-\omega_i(x) \frac{du}{dx} \Big|_a^b + \int_a^b \frac{d\omega_i(x)}{dx} \frac{du}{dx} dx = \int_a^b \omega_i(x) f(x) dx \quad (2.3)$$

This is known as the weak form of the differential equation. In equation (2.3), we only require that $u, \omega_i \in H^1(a,b)$, once differentiable. So linear representation of such functions is allowed. Also since the integrals can be broken into subintervals, piecewise functions are

$j = 0, N + 1$ and f is load vector consisting of

$$f_i = \int_{x_{j-1}}^{x_{j+1}} \varphi_i(x) f(x) dx \quad (2.10)$$

The first and last equations are special with basis functions which are "half hats" and are

$$\sum_{j=0}^1 U_j \int_{x_0}^{x_1} d\varphi_0$$

First and last equations reduced to

$$-\frac{U_1 - U_0}{x_1 - x_0} = \int_{x_0}^{x_1} \varphi_0(x) f(x) dx, (i = 0) \quad (2.16)$$

and

$$\frac{U_{N+1} - U_N}{x_{N+1} - x_N} = \int_{x_N}^{x_{N+1}} \varphi_N(x) f(x) dx, (i = N + 1) \quad (2.17)$$

The right hand side integrals can be evaluated by numerical integration.

The stiffness matrix K is singular, which follows from the fact that the $\varphi_i(x)$ form a partition of unity, so

$$\sum_{i=0}^{N+1} \varphi_i(x) = 1 \quad \sum_{i=0}^{N+1} \frac{d\varphi_i}{dx} = 0 \quad (2.18)$$

It means all column sums of the determinant of Matrix K are zero. By applying boundary conditions, it is possible to invert this matrix.

2.1.4 Evaluation of Stiffness matrix K and Load Vector f for Quadratics

In this section we explain how can we construct solution by adding an extra node in each element as p-refinement.

There are deficiencies in linear finite elements formulation in convection-diffusion problems. In such situations to examine the numerical solution of 1D convection-diffusion problem discretise with quadratic shape functions as shown in figure (2.2).

First of all we establish a matrix equation for the given problem, as for linear finite elements, then the discrete solution of the problem is analysed.

As shown in figure (2.2), we consider a generic element with nodes 1, 2 and 3, where node 2 is a mid-side. With reference to the condition $0 < x < 1$, the shape function of the element are

$$N_1(x) = 2(x - \frac{1}{2})(x - 1), \quad N_2(x) = -4x(x - 1), \quad \text{and} \quad N_3(x) = 2x(x - \frac{1}{2}).$$

We can establish an element stiffness matrix K^e of the quadratic elements [1] as follows

$$K^e = \int_{\Omega} \begin{pmatrix} \frac{\partial N_1}{\partial x} \frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial x} \frac{\partial N_2}{\partial x} & \frac{\partial N_1}{\partial x} \frac{\partial N_3}{\partial x} \\ \frac{\partial N_2}{\partial x} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} \frac{\partial N_2}{\partial x} & \frac{\partial N_2}{\partial x} \frac{\partial N_3}{\partial x} \\ \frac{\partial N_3}{\partial x} \frac{\partial N_1}{\partial x} & \frac{\partial N_3}{\partial x} \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \frac{\partial N_3}{\partial x} \end{pmatrix} dx$$

and load vector $\underline{f}$ as follows

$$f_i = \int_{\Omega} f(x) N_i(x) dx$$

where $i = (1, 2, 3)$.

2.2 A More General first order Differential Equation

2.2.2 Quadratic Finite Elements

Similarly from (2.19), apply quadratic finite elements and after applying all the calculations, we get the matrix equation

$$K \underline{U} = \underline{f}$$

where K is the assembly of element matrices

$$K_{ij}^e = \int_{\Omega} p(x) \begin{pmatrix} \frac{\partial N_1}{\partial x} \frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial x} \frac{\partial N_2}{\partial x} & \frac{\partial N_1}{\partial x} \frac{\partial N_3}{\partial x} \\ \frac{\partial N_2}{\partial x} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} \frac{\partial N_2}{\partial x} & \frac{\partial N_2}{\partial x} \frac{\partial N_3}{\partial x} \\ \frac{\partial N_3}{\partial x} \frac{\partial N_1}{\partial x} & \frac{\partial N_3}{\partial x} \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \frac{\partial N_3}{\partial x} \end{pmatrix} dx$$

and

$$f_i = \int_0^1 f(x) N_i(x) dx$$

is a load vector.

Chapter 3

dy

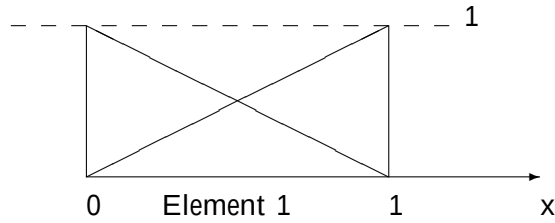


Figure 3.2: linear finite elements for one element

and load vectors for $f(x) = x^2$ are

$$f_0 = \int_0^1 (1-x)x^2 dx = \frac{1}{12}$$

$$f_1 = \int_0^1 x^3 dx = \frac{1}{4}$$

By solving the test problem, we get approximate value of velocity potential U shown in table (3.1).

Table 3.1: Results for equation $\frac{d^2 u}{dx^2} = x$ and x^2 for one Element.

x value	Exact $f(x)=x$	LFE $f(x)=x$	Error (u-U)	Exact $f(x)=x^2$	LFE $f(x)=x^2$	error (u-U)
0	-1.33333	-1.16667	-0.166667	-1.08333	-1.5	0.416667
1	0	0	0	0	0	0

- Stiffness Matrix and load vector for Two element

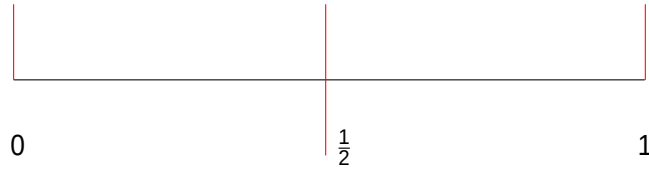


Figure 3.3: Linear finite element for Two elements.

$$f_1 = \int_0^{\frac{1}{2}} (2x)x^2 dx + \int_{\frac{1}{2}}^1 (2 - 2x)x^2 dx = \frac{7}{48}$$

and

$$f_2 = \int_{\frac{1}{2}}^1 (2x - 1)x^2 dx = \frac{17}{96}$$

Similarly if $f(x) = x$ the load vector is

$$f_i = \left(\frac{1}{24} \quad \frac{1}{4} \quad \frac{5}{24} \right)^T$$

The solution for U is shown in table (3.2)

Table 3.2: Results for equation $\frac{d^2 u}{dx^2} = x$ and x^2 of Two Element.

x value	Exact $f(x)=x$	LFE $f(x)=x$	Error (u-U)	Exact $f(x)=x^2$	LFE $f(x)=x^2$	error (u-U)
0	-1.33333	-1.16667	-0.166667	-1.08333	-1.08333	2.22045e-016
0.5	-0.791667	-0.645833	-0.145833	-0.578125	-0.578125	1.11022e-016
1	0	0	0	0	0	0

•

and the load vector is

$$f_0 = \int_0^{\frac{1}{4}} (1 - 4x)x^2 dx = \frac{1}{768}$$

,

$$f_1 = \int_0^{\frac{1}{4}} (4x)x^2 dx + \int_{\frac{1}{4}}^{\frac{1}{2}} (2 - 4x)x^2 dx = \frac{7}{384}$$

$$f_2 = \int_{\frac{1}{4}}^{\frac{1}{2}} (4x - 1)x^2 dx + \int_{\frac{1}{2}}^{\frac{3}{4}} (3 - 4x)x^2 dx = \frac{25}{384}$$

$$f_3 = \int_{\frac{1}{2}}^{\frac{3}{4}} (4x - 2)x^2 dx + \int_{\frac{3}{4}}^1 (4 - 4x)x^2 dx = \frac{55}{384}$$

and

$$f_4 = \int_{\frac{3}{4}}^1 (4x - 3)x^2 dx = \frac{27}{256}$$

The load vector f is when $f(x) = x$ is

$$f_i = \left(\frac{1}{96} \quad \frac{1}{16} \quad \frac{1}{8} \quad \frac{3}{8} \quad \frac{11}{96} \right)^T$$

Table 3.3: Results for equation d^2u

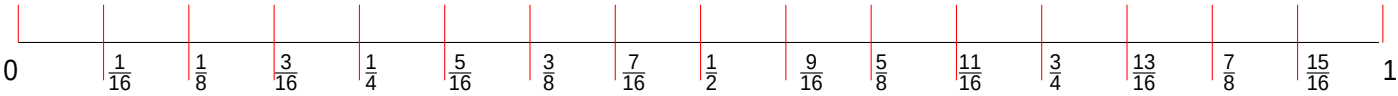


Figure 3.6: Linear finite element for sixteen elements.

Table 3.5: Results for equation $\frac{d^2u}{dx^2} = x^2$ of Sixteen Element.

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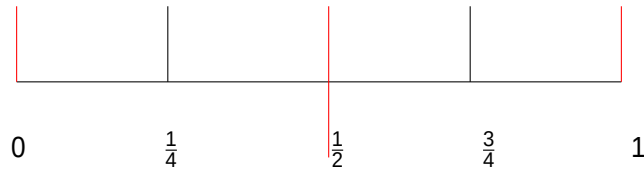


Figure 3.7: Quadratic finite element for Two elements.

$$N_1(x) = 8(x - \frac{3}{4})($$

$$N_2(x) = -64(x - \frac{1}{2})(x - \frac{3}{4}),$$

$$N_3(x) = 32(x - \frac{1}{2})(x - \frac{5}{8})$$

4. Fourth Element

$$N_1(x) = 32(x - \frac{7}{8})(x - 1),$$

$$N_2(x) = -64(x - \frac{3}{4})(x - 1),$$

$$N_3(x) = 32(x - \frac{3}{4})(x - \frac{7}{8})$$

For each element, the stiffness matrix is

$$K^e = \begin{pmatrix} \frac{28}{3} & \frac{-32}{3} & \frac{4}{3} \\ \frac{-32}{3} & \frac{64}{3} & \frac{-32}{3} \\ \frac{4}{3} & \frac{-32}{3} & \frac{28}{3} \end{pmatrix}$$

and load vectors for $f(x) = x$ and $f(x) = x^2$ are

$$f_i^1 = \left[0 \quad \frac{1}{48} \quad \frac{1}{96} \quad \frac{1}{16} \quad \frac{1}{48} \quad \frac{5}{48} \quad \frac{1}{32} \quad \frac{7}{48} \quad \frac{1}{24} \right]^T$$

and

$$f_i^2 = \left[\frac{-1}{3840} \quad \frac{1}{320} \quad \frac{3}{1280} \quad \frac{23}{960} \quad \frac{21}{320} \quad \frac{89}{3540} \quad \frac{41}{320} \quad \frac{53}{1280} \right]^T$$

The matrix after assembly is

$$K = \frac{1}{3} \begin{pmatrix} 28 & -32 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ -32 & 64 & -32 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & -32 & 56 & -32 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & -32 & 64 & -32 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -32 & 56 & -32 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -32 & 64 & -32 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & -32 & 56 & -32 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -32 & 64 & -32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & -32 & 28 \end{pmatrix}$$

Table (3.7) gives the solution for four elements

and

$$f_i^2 = \begin{bmatrix} \frac{-1}{30720} & \frac{1}{2560} & \frac{3}{10240} & \frac{23}{7680} & \frac{3}{10240} & \frac{21}{2560} & \frac{89}{30720} & \frac{41}{2560} & \frac{53}{10240} & \frac{203}{7680} & \frac{83}{10240} \\ \frac{101}{2560} & \frac{359}{30720} & \frac{141}{2560} & \frac{163}{10240} & \frac{563}{7680} & \frac{213}{10240} & & & & & \\ & & & & & & & & & & \end{bmatrix}^T$$

Table 3.8: Results for equation $\frac{d^2 u}{dx^2} = x$ and x^2 for eight Element.

x	Exact (x)	QFE (x)	Error (x)	Exact (x^2)	QFE (x^2)	Error (x^2)
0	-1.33333	-1.13688	-0.196452	-1.08333	-1.06904	-0.0142904
0.0625	-1.27075	-1.07434	-0.196411	-1.02083	-1.00654	-0.0142901
0.125	-1.20768	-1.01156	-0.196126	-0.958313	-0.944023	-0.0142904
0.1875	-1.14364	-0.948446	-0.19519	-0.89573	-0.881459	-0.0142718
0.25	-1.07813	-0.884603	-0.193522	-0.833008	-0.818754	-0.0142537
0.3125	-1.01066	-0.820109	-0.190552	-0.770039	-0.755944	-0.0140948
0.375	-0.940755	-0.754395	-0.186361	-0.706685	-0.692749	-0.0139364
0.4375	-0.86792	-0.687703	-0.180216	-0.64278	-0.629184	-0.0135963
0.5	-0.791667	-0.619303	-0.172363	-0.578125	-0.564868	-0.0132568
0.5625	-0.711507	-0.549601	-0.161906	-0.512491	-0.499897	-0.0125933
0.625	-0.626953	-0.477702	-0.149251	-0.445618	-0.433687	-0.0119303
0.6875	-0.537516	-0.404175	-0.133341	-0.377216	-0.366456	-0.0107602
0.75	-0.442708	-0.327962	-0.114746	-0.306966	-0.297375	-0.00959066
0.8125	-0.342041	-0.250061	-0.09198	-0.234516	-0.226826	-0.00769018
0.875	-0.235026	-0.170573	-0.0644531	-0.159485	-0.153695	-0.0057902
0.9375	-0.121175	-0.0857747	-0.0354004	-0.0814603	-0.0785655	-0.00289485
1	0	0	0	0	0	0

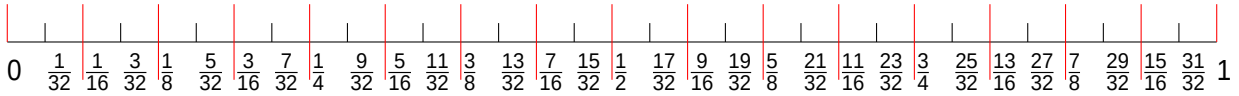


Figure 3.10: Quadratic finite element for sixteen elements.

For each element, the stiffness matrix is

$$K^e = \begin{pmatrix} \frac{112}{3} & \frac{-128}{3} & \frac{16}{3} \\ \frac{-128}{3} & \frac{256}{3} & \frac{-128}{3} \\ \frac{16}{3} & \frac{-128}{3} & \frac{112}{3} \end{pmatrix}$$

and load vectors for f_i^2

3.3 A More General 1-D Differential Equation

Now consider the problem

$$-\frac{d}{dx} \left((2x+1) \frac{du}{dx} \right) = x^2 \quad (3.3)$$

in $(0, 1)$ with artificial boundary conditions $\frac{du}{dx} = 1$ at $x = 0$ and $u = 0$ at $x = 1$. This is representative of the moving mesh movement equation.

3.3.1 Linear Finite Elements

So by using the approach discussed in chapter 2, we can find stiffness and load vector as shown below.

- Solution for One element

Stiffness matrix for one element with $h = 1$ and $p(x) = 2x + 1$ is

$$K = \frac{1}{h^2} \int_0^1 p(x) \frac{d\phi_i}{dx} d\phi$$

$$K_2^e = \frac{1}{h^2} \int_{\frac{1}{2}}^1 p(x) \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} dx = \begin{pmatrix} 5 & -5 \\ -5 & 5 \end{pmatrix}$$

and load vectors are

$$f_0 = \int_0^{\frac{1}{2}} (1 - 2x)x^2 dx = \frac{1}{96}$$

$$f_1 = \int_0^{\frac{1}{2}} 2x^3 dx + \int_{\frac{1}{2}}^1 (2 - 2x)x^2 dx = \frac{7}{48}$$

$$f_2 = \int_{\frac{1}{2}}^1 (2x - 1)x^2 dx = \frac{7}{48}$$

$$K_3^e = \frac{1}{h^2} \int_{\frac{1}{2}}^{\frac{3}{4}} p(x) \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} dx = \begin{pmatrix} 9 & -9 \\ -9 & 9 \end{pmatrix}$$

$$K_4^e = \frac{1}{h^2} \int_{\frac{3}{4}}^1 p(x) \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} dx = \begin{pmatrix} 11 & -11 \\ -11 & 11 \end{pmatrix}$$

and load vectors are

$$f_0 = \int_0^{\frac{1}{4}} (1 - 4x)x^2 dx = \frac{1}{768}$$

$$f_1 = \int_0^{\frac{1}{4}} 4x^3 dx + \int_{\frac{1}{4}}^{\frac{1}{2}} (2 - 4x)x^2 dx = \frac{7}{384}$$

$$f_2 = \int_{\frac{1}{4}}^{\frac{1}{2}} (4x - 1)x^2 dx + \int_{\frac{1}{2}}^{\frac{3}{4}} (3 - 4x)x^2 dx = \frac{25}{384}$$

$$f_3 = \int_{\frac{1}{2}}^{\frac{3}{4}} (4x - 2)x^2 dx + \int_{\frac{3}{4}}^1 (4 - 4x)x^2 dx = \frac{55}{384}$$

$$f_4 = \int_{\frac{3}{4}}^1 (4x - 3)x^2 dx = \frac{27}{256}$$

The stiffness matrix for four elements is

$$K = \begin{pmatrix} 5 & -5 & 0 & 0 & 0 \\ -5 & 12 & -7 & 0 & 0 \\ 0 & -7 & 16 & -9 & 0 \\ 0 & 0 & -9 & 20 & -11 \\ 0 & 0 & 0 & -11 & 11 \end{pmatrix}$$

Table 3.12: Results for equation $-\frac{d}{dx}((2x+1)\frac{du}{dx}) = x^2$ for Four Element.

x values	Exact values (u)	Linear Finite values (U)	Error (u-U)
0	-0.516638	-0.511708	-0.00493015
0.25	-0.314139	-0.311968	-0.00217054
0.5	-0.172985	-0.171901	-0.00108365
0.75	-0.0706532	-0.0701941	-0.000459115
1	0	0	0

Similarly we can find solutions for eight and sixteen elements.

We get the solution as shown in Table (3.14)

Table 3.14: Results for equation $-\frac{d}{dx}((2x+1)\frac{du}{dx}) = x^2$ for Four Element.

x values	Exact values (u)	Linear Finite values (U)	Error (u-U)
0	-0.516638	-0.521588	0.00495024
0.125	-0.405083	-0.410112	0.00502911
0.25	-0.314139	-0.319118	0.00497899
0.375	-0.237867	-0.24268	0.00481295
0.5	-0.172985	-0.177631	0.00464633
0.625	-0.117608	-0.12152	0.00391259
0.75	-0.0706532	-0.0739058	0.00325257
0.875	-0.0315377	-0.0330893	0.00155158
1	0	0	0

Similarly we can find the solution for eight and sixteen elements.

Chapter 4

Recovery of the Solution to the first order Problem

In this chapter we describe how we can accomplish the solution of first order differential equation i.e. recovery of velocity. In previous chapter we mentioned the approach that replaced the velocity (y) with the velocity potential (u) as $y = \frac{du}{dx}$ and solved it for U . Now we describe the approach that gives us Y from U which is an approximation to the exact solution.

4.1 Discontinuous Solution (by differentiation)

- Linears

We can apply different approaches to find the approximated velocity vector (Y). One way is to get the Y values from U for each element as follows

$$Y_i = \frac{du}{dx} = \frac{U_{i+1} - U_i}{x_{i+1} - x_i}$$

Here U is piecewise linear and the Y function is piecewise constant. So Y is not continuous. We need to look for another way to go from U to Y ($=dU_{\overline{dx}}$) which gives us a continuous function which we also discuss in the next section.

- Quadratics

In this case recovery of Y from U by differentiation also gives us discontinuous func-

tion. So we need to adapt these values for each node to get continuous function which we discuss in next section.

4.2 Continuous Solution

Consider the following method to find the continuous solution for linear finite elements,

4.2.1 Least Squares for Linears

Let's consider the following least square approach

$$\left\| y - \frac{dU}{dx} \right\|^2 = \int_0^1 \left(Y - \frac{dU}{dx} \right)^2 dx$$

Minimise it over Y , where $Y = \sum \bar{y}_j \phi_j$ is continuous, requiring minimization of

$$\int_0^1 \left(\sum_{j=0}^1 \bar{y}_j \phi_j - \frac{dU}{dx} \right)^2 dx$$

. Minimise over $\bar{y}$ values,

$$\frac{d}{d\bar{y}_i} \left(\int_0^1 \sum_{j=0}^1 \bar{y}_j \phi_j - \frac{dU}{dx} \right)^2 dx = 0$$

and generally, it is

$$\int_0^1 \left(\sum_j \bar{y}_j \phi_j - \frac{dU}{dx} \right) \phi_i dx = 0$$

The above equation can be written in the form

$$\sum_j \left(\int_0^1 \phi_i \phi_j dx \right) \bar{y}_j - \int_0^1 \phi_i \frac{dU}{dx} dx = 0$$

and in matrix form as

$$M \underline{\bar{y}} = \underline{g}$$

where M is the element mass matrix. For one element $M^e = h \begin{pmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{pmatrix}$, $\bar{y}$ is a velocity vector and

$$g_i^e = \frac{dU}{dx} \int_0^1 \varphi_i dx = \frac{1}{2} h \frac{dU}{dx} = \begin{pmatrix} \frac{1}{2} \left(\frac{U_1 - U_0}{x_1 - x_0} \right) \\ \frac{1}{2} \left(\frac{U_{i+1} - U_i}{x_{i+1} - x_i} \right) \\ \dots \\ \frac{1}{2} \left(\frac{U_n - U_{n-1}}{x_n - x_{n-1}} \right) \end{pmatrix}$$

We can get an approximation to the velocity vector by applying the approach discussed above. There are some solutions in section (4.3) for the test problems discussed in chapter 3.

4.2.2 For Quadratics

After finding the values of $\underline{U}$

Table 4.1: Linear finite elements to solve equation $\frac{dy}{dx} = x$ and x^2 for 2 elements.

x	y=Exact(f(x)=x)	Y=LFE(f(x)=x)	Error(x)	y=Exact (f(x)=x ²)	Y=LFE(f(x)=x ²)	Error(x ²)
0	1	0.973958	0.0260417	1	0.973958	0.0260417
0.5	1.125	1.08333	0.0416667	1.04167	1.08333	-0.0416667
1	1.5	1.19271	0.307292	1.33333	1.19271	0.140625

Table 4.2: Linear finite elements to solve equation $\frac{dy}{dx} = x$ and x^2 for 4 elements.

x	y=Exact(f(x)=x)	Y=LFE(f(x)=x)	Error(x)	y=Exact (f(x)=x ²)	Y=LFE(f(x)=x ²)	Error(x ²)
0	1	0.998512	0.0014881	1	0.998512	0.0014881
0.25	1.03125	1.00688	0.0243676	1.00521	1.00688	-0.00167411
0.5	1.125	1.03646	0.0885417	1.04167	1.03646	0.00520833
0.75	1.28125	1.15978	0.121466	1.14063	1.15978	-0.0191592
1	1.5	1.2619	0.238095	1.33333	1.2619	0.0714286

Table 4.3: Linear finite elements to solve equation $\frac{dy}{dx} = x$ and x^2 for 8 elements.

x	y=Exact(f(x)=x)	Y=LFE(f(x)=x)	Error(x)	y=Exact (f(x)=x ²)	Y=LFE(f(x)=x ²)	Error(x ²)
0	1	0.999904	9.58824e-005	1	0.999904	9.58824e-005
0.125	1.00781	1.00068	0.00713245	1.00065	1.00068	-2.90044e-005
0.25	1.03125	1.00519	0.0260618	1.00521	1.00519	2.01353e-005
0.375	1.07031	1.01763	0.0526828	1.01758	1.01763	-5.15368e-005
0.5	1.125	1.04148	0.0835193	1.04167	1.04148	0.000186012
0.625	1.19531	1.08207	0.11324	1.08138	1.08207	-0.000692511
0.75	1.28125	1.13804	0.143209	1.14063	1.13804	0.00258403
0.875	1.38281	1.23295	0.149862	1.22331	1.23295	-0.00964361
1	1.5	1.29734	0.202657	1.33333	1.29734	0.0359904

Table 4.4: Linear finite elements to solve equation $\frac{dy}{dx} = x$ and x^2 for 16 elements.

x	y=Exact(f(x)=x)	Y=LFE(f(x)=x)	Error(x)	y=Exact (f(x)=x ²)	Y=LFE(f(x)=x ²)	Error(x ²)
0	1	2.366	-1.366	1	2.366	-1.366
0.0625	1.00195	0.634063	0.36789	1.00008	0.634063	0.366018
0.125	1.00781	1.09873	-0.0909128	1.00065	1.09873	-0.0980742
0.1875	1.01758	0.975918	0.0416598	1.0022	0.975918	0.0262789
0.25	1.03125	1.01225	0.0190003	1.00521	1.01225	-0.00704141
0.3125	1.04883	1.00829	0.0405423	1.01017	1.00829	0.00188673
0.375	1.07031	1.01808	0.0522289	1.01758	1.01808	-0.000505516
0.4375	1.0957	1.02778	0.067925	1.02791	1.02778	0.000135334
0.5	1.125	1.0417	0.0832975	1.04167	1.0417	-3.58178e-005
0.5625	1.1582	1.05932	0.0988849	1.05933	1.05932	7.9377e-006
0.625	1.19531	1.08138	0.113936	1.08138	1.08138	4.067e-006
0.6875	1.23633	1.10834	0.127987	1.10832	1.10834	-2.42057e-005
0.75	1.28125	1.14053	0.140718	1.14063	1.14053	9.27558e-005
0.8125	1.33008	1.17914	0.150939	1.17879	1.17914	-0.000346818
0.875	1.38281	1.22201	0.1608	1.22331	1.22201	0.00129451
0.9375	1.43945	1.27949	0.159964	1.27466	1.27949	-0.00483124
1	1.5	1.3153	0.184697	1.33333	1.3153	0.0180304

2. Linear Solution for Second Test problem

We fixed the error occurred in finding the solution of the second test problem by applying linear finite elements for moving mesh. So the following tables (4.5), (4.6), (4.7) and (4.8) show the results for exact velocity y and approximated velocity Y recovered from the velocity potential U . All the tables are self explanatory, showing results for 2, 4, 8, and 16 elements.

Table 4.5: Results by solving equation $-\frac{d}{dx}((2x+1)y) = x^2$ for 2 elements.

x values	Exact values (y)	Linear values (Y)	Error(y-Y)
0	1	0.740278	0.259722
0.5	0.479167	0.498611	-0.019444
1	0.222222	0.256944	-0.034722

Table 4.6: Results by solving equation $-\frac{d}{dx}((2x+1)y) = x^2$ for 4 elements.

x values	Exact values (y)	Linear values (Y)	Error(y-Y)
0	1	0.854184	0.145816
0.25	0.663194	0.688506	-0.025312
0.5	0.479167	0.469468	0.009698
0.75	0.34375	0.334909	0.008841
1	0.222222	0.25371	-0.031488

Table 4.7: Results by solving equation $-\frac{d}{dx}((2x+1)y) = x^2$ for 8 elements.

x values	Exact values (y)	Linear values (Y)	Error(y-Y)
0	1	0.925543	0.074457
0.125	0.799479	0.815147	-0.015668
0.25	0.663194	0.656595	0.006599
0.375	0.561384	0.561588	-0.000204
0.5	0.479167	0.478101	0.001066
0.625	0.408275	0.407409	0.000866
0.75	0.34375	0.344444	-0.000694
0.875	0.282434	0.277335	0.005099
1	0.222222	0.239135	-0.016913

Table 4.10: Results by solving equation $\frac{dy}{dx}$ with $f(x) = x$ and $f(x) = x^2$ for 4 Elements.

x	y=Exact (f(x)=x)	Y=QFE (f(x)=x)	Error (x)	y=Exact (f(x)=x ²)	Y=QFE (f(x)=x ²)	Error (x ²)
0	1	0.994792	0.00520833	1	0.998958	0.00104167
0.125	1.00781	1.01042	-0.00260417	1.00065	1.0013	-0.000651042
0.25	1.03125	1.02604	0.00520833	1.00521	1.00365	0.0015625
0.375	1.07031	1.0625	0.0078125	1.01758	1.01562	0.00195313

2. Quadratic Solution for Second Test Problem

The following tables (4.13), (4.14), (4.15) and (4.16) give us the results for 2, 4, 8 and 16 elements by using quadratic approach.

Table 4.15: Results by solving equation $-\frac{d}{dx}((2x+1)y) = x^2$ for 8 elements.

x values	Exact values (y)	Quadratic values (Y)	Error(y-Y)
0	1	0.991845	0.0081547
0.0625	0.888817	0.892426	-0.00360984
0.125	0.799479	0.793007	0.00647168
0.1875	0.725675	0.727765	-0.00208987
0.25	0.663194	0.659971	0.0032238
0.3125	0.609125	0.61174	-0.00261577
0.375	0.561384	0.560897	0.000486572
0.4375	0.518446	0.521957	-0.00351112
0.5	0.479167	0.480754	-0.00158723
0.5625	0.44267	0.448019	-0.00534887
0.625	0.408275	0.412336	-0.00406035
0.6875	0.375445	0.383522	-0.00807682
0.75	0.34375	0.350878	-0.007128
0.8125	0.312841	0.324515	-0.0116744
0.875	0.282434	0.293326	-0.010892
0.9375	0.252293	0.268426	-0.0161336
1	0.222222	0.237632	-0.0154098

Table 4.16: Results by solving equation $-\frac{d}{dx}((2x+1)y) = x^2$ for 16 elements.

x values	Exact values (y)	Quadratic values (Y)	Error(y-Y)
0	1	0.997705	0.00229458
0.03125	0.941167	0.942245	-0.00107789
0.0625	0.888817	0.886784	0.0020324
0.09375	0.841874	0.842661	-0.000787368
0.125	0.799479	0.798074	0.00140485
0.15625	0.760936	0.761631	-0.000695042
0.1875	0.725675	0.724777	0.000897804
0.21875	0.693225	0.694016	-0.000790709
0.25	0.663194	0.662797	0.000397904
0.28125	0.635254	0.636327	-0.00107276
0.3125	0.609125	0.609288	-0.000163062
0.34375	0.584569	0.586112	-0.00154289
0.375	0.561384	0.562212	-0.000828374
0.40625	0.539394	0.541598	-0.0022039
0.4375	0.518446	0.520073	-0.00162671
0.46875	0.498409	0.501468	-0.00305881
0.5	0.479167	0.481744	-0.00257778
0.53125	0.460617	0.464727	-0.00411046
0.5625	0.44267	0.446366	-0.00369557
0.59375	0.425246	0.430608	-0.00536141
0.625	0.408275	0.413266	-0.00499026
0.65625	0.391694	0.398508	-0.00681386
0.6875	0.375445	0.381915	-0.00646945
0.71875	0.359479	0.367949	-0.00846977
0.75	0.34375	0.351889	-0.00813891
0.78125	0.328216	0.338547	-0.0103308
0.8125	0.312841	0.322844	-0.0100031
0.84375	0.29759	0.309989	-0.0123983
0.875	0.282434	0.294499	-0.0120656
0.90625	0.267343	0.282017	-0.0146735
0.9375	0.252293	0.266622	-0.0143291
0.96875	0.23726	0.254417	-0.0171575
1	0.222222	0.239018	-0.0167959

Chapter 5

Discussion

Comparison of the results.

5.1 Exact solution for y

- First Problem

To find the exact solution of

dy

$$u = \frac{x^3}{3} + x - \frac{4}{3} \quad (5.4)$$

the exact solution for u .

Let us consider $f(x) = x^2$, giving

$$\frac{du^2}{dx^2} = x^2 \quad (5.5)$$

Now by integrating (5.5), the exact solution for y is

$$y = \frac{x^3}{3} + 1 \quad (5.6)$$

and for u is

$$u = \frac{x^4}{12} + x - \frac{13}{12} \quad (5.7)$$

•

5.2 Results

5.2.1 Linear and Quadratic continuous Solution for y (Problem-1)

All the graphs shown below give us enough information to understand about the results. By comparison of linear and quadratic results it is clear that linear approach is not good as compared to quadratic in case of first test problem. Quadratic finite elements gives us better results.

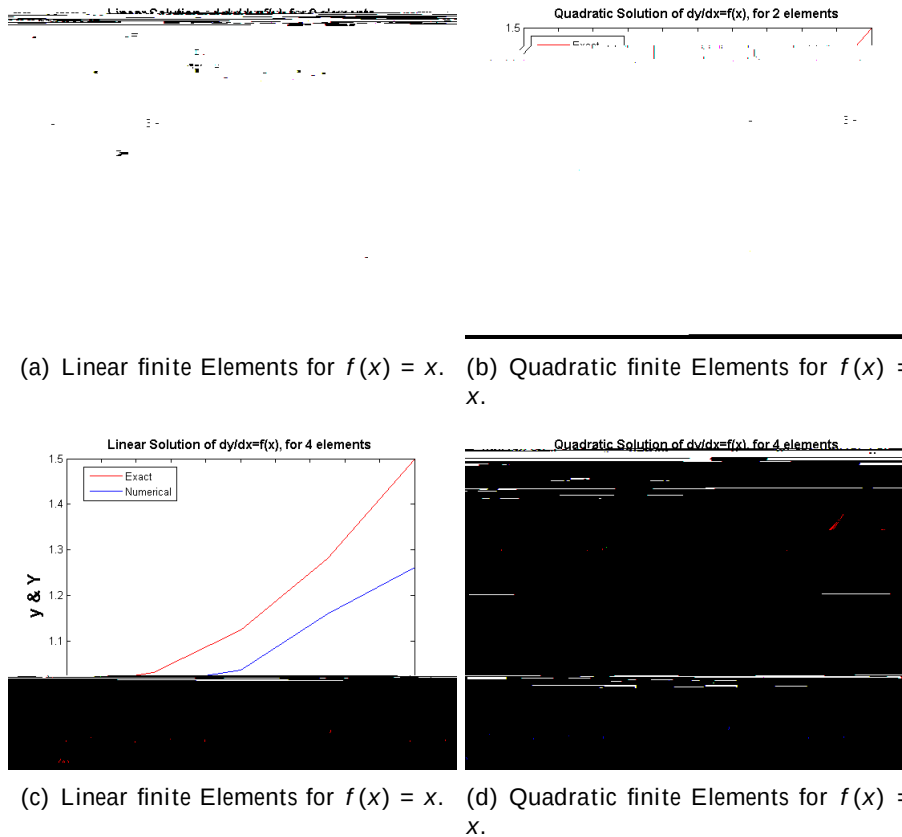


Figure 5.1: Graphs showing the results for linear and quadratic finite elements of first test problem for 2 and 4 elemets.

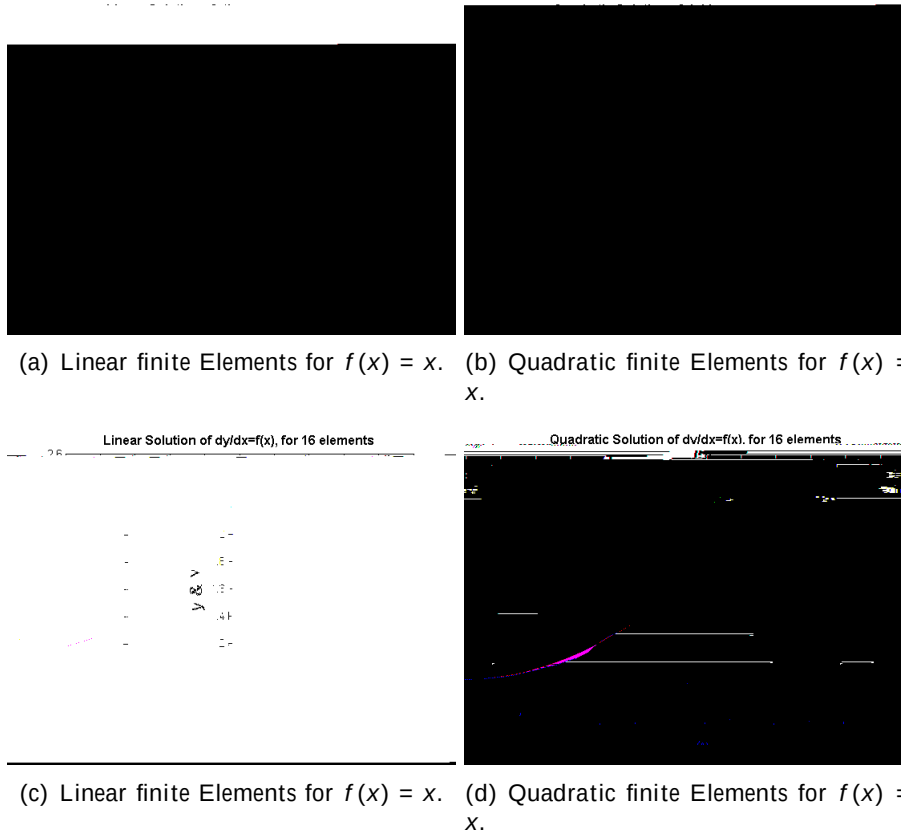


Figure 5.2: Graphs showing the results for linear and quadratic finite elements of first test problem for 8 and 16 elements.

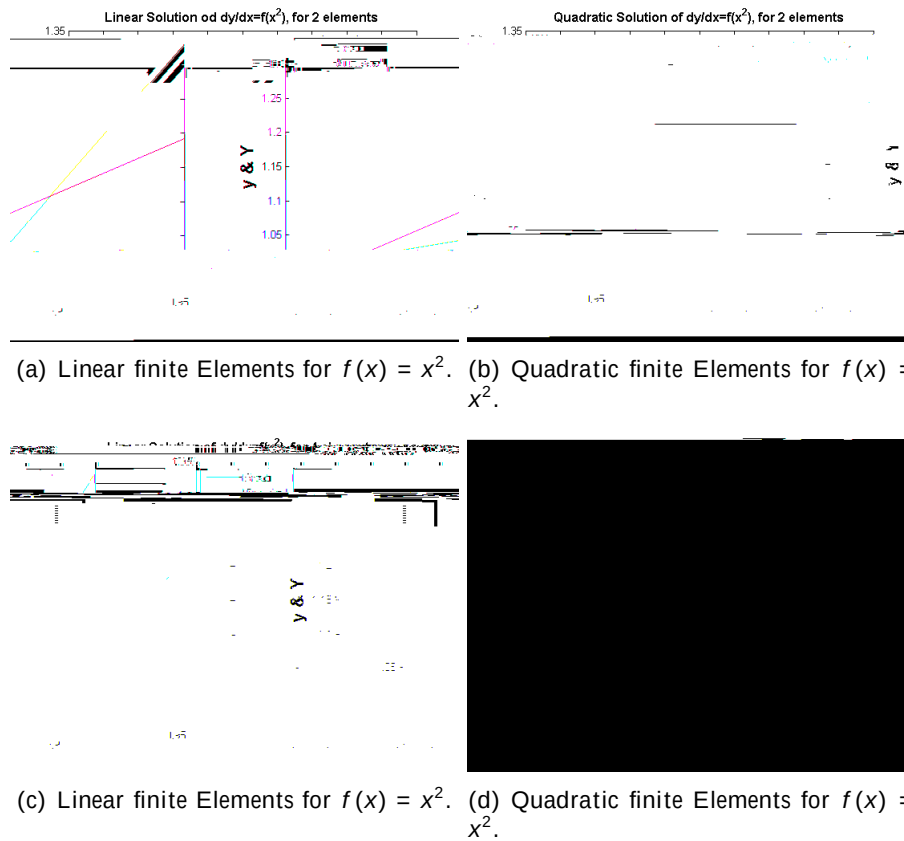
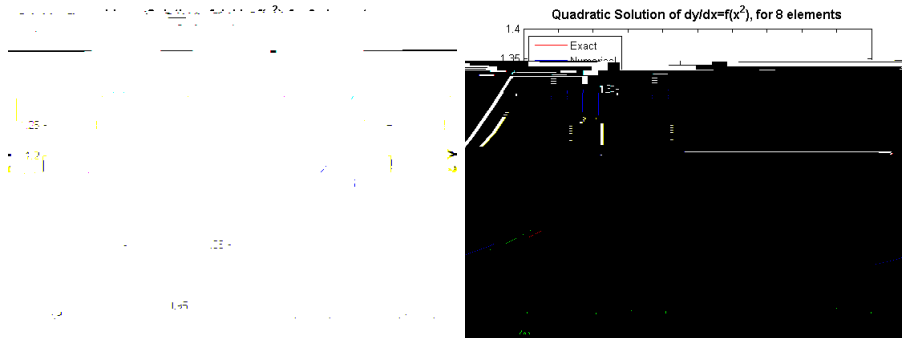
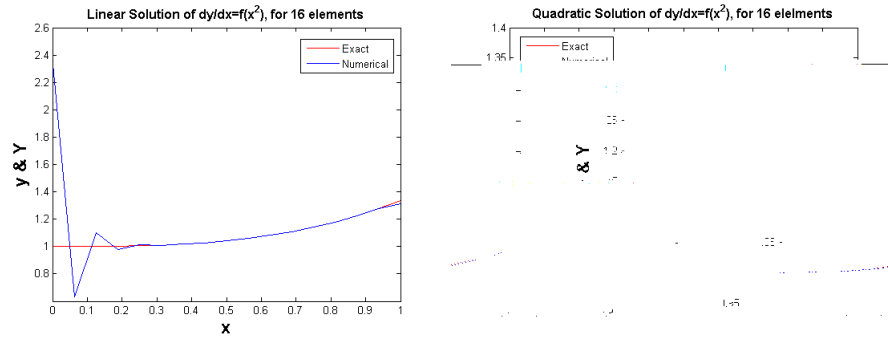


Figure 5.3: Graphs showing the results for linear and quadratic finite elements of first test problem for 2 and 4 elements.



(a) Linear finite Elements for $f(x) = x^2$. (b) Quadratic finite Elements for $f(x) = x^2$.



(c) Linear finite Elements for $f(x) = x^2$. (d) Quadratic finite Elements for $f(x) = x^2$.

5.2.2 Linear and Quadratic continuous Solution for y (Problem-2)

The comparison of the results for linear and quadratic finite elements of second test problem tells us

- Graphs show that numerical values for Y get better as we increase the number of elements.
- Linear results are really good except for end values for 16 elements.
- Quadratic results are better at the start of any number of elements.

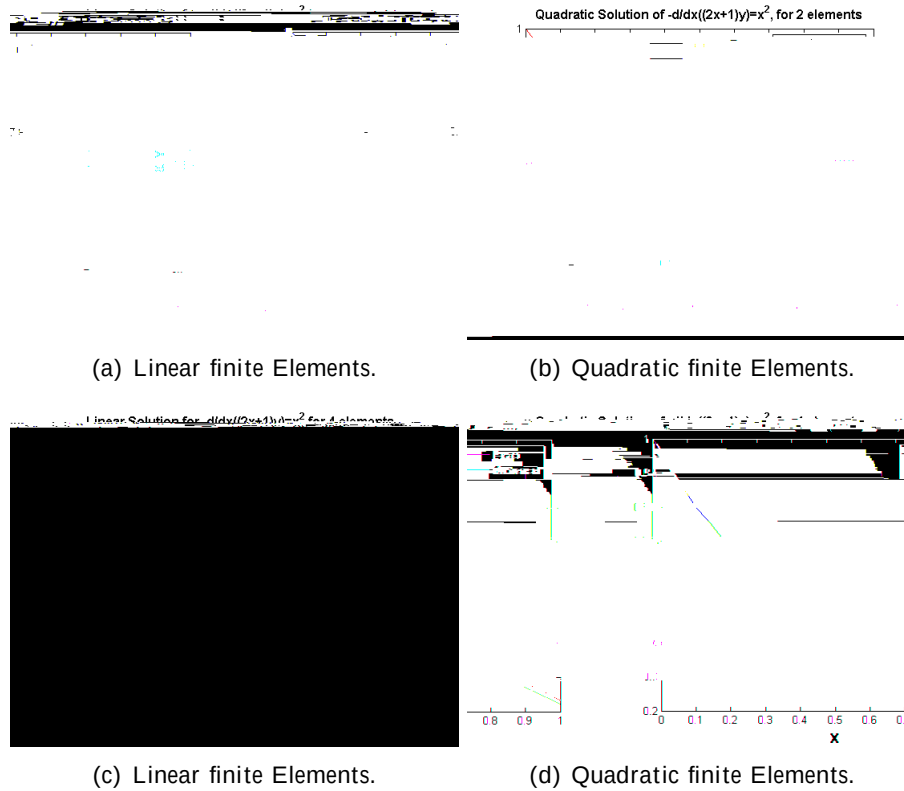


Figure 5.5: Graphs showing the results for linear and quadratic finite elements of second test problem for 2 and 4 elements.

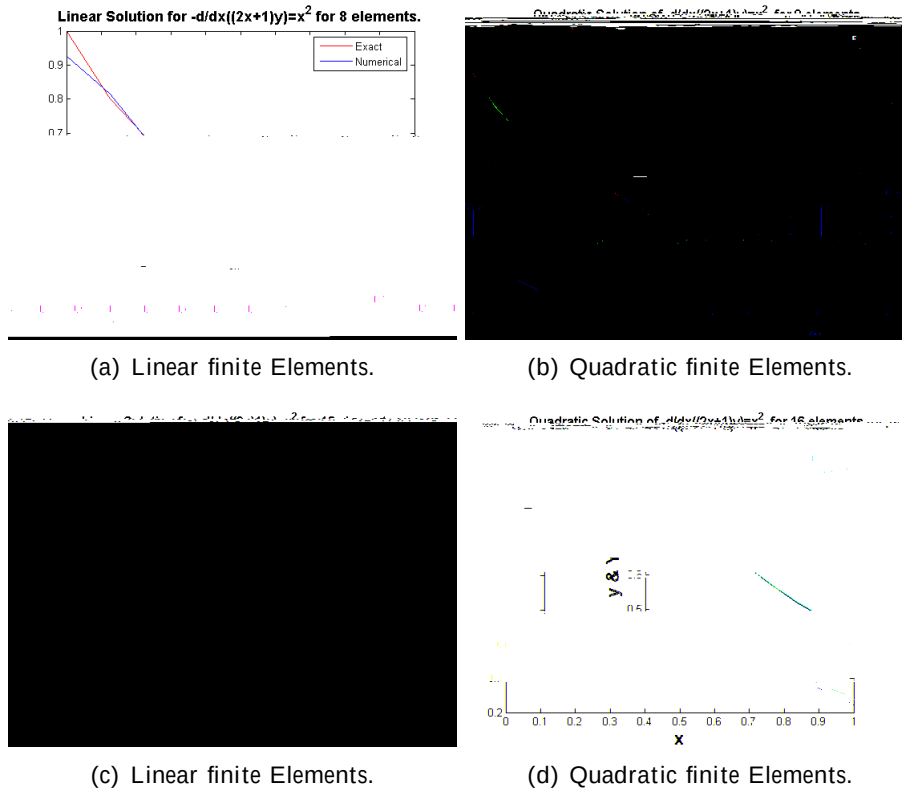


Figure 5.6: Graphs showing the results for linear and quadratic finite elements of second test problem for 8 and 16 elements.

5.3 Conclusion

We have showed that when linear elements approach does not work very well to recover the values of velocity (Y)

ential equations as well as the Sturm-Liouville type differential equations. In this chapter we solved test problems to investigate the numerical results.

Chapter Four introduced the results for moving boundary and discussed the possible behaviour that can arise as the boundary moves. We also discussed the numerical results of the test problem and compared them with the exact solutions to investigate the errors.

5.4 Future Work

Our next target is to find the solutions for higher order differential equations.

Bibliography

[1]