Boundary integral methods in high frequency scattering

A stract

In this article we review recent progress on the design, analysis and implementation of numerical-asymptotic boundary integral methods for the computation of frequencydomain acoustic scattering in a homogeneous unbounded medium by a bounded obstacle. The main aim of the methods is to allow computation of scattering at arbitrarily high frequency with finite computational resources.

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and **u/n** oder $\sin d$ and $\sin d$ ondary \sin methods \cos methods approximate the whole \sqrt{p} oscillatory) **o**u/∂n by \sqrt{p} c^{ort} the by contrast the hybrid methods which we shall be sha dic in the following section \mathbf{R} over \mathbf{y} obtain an analytic information about the oscillations in \mathbf{u}/\mathbf{n} in this information is the numerical is the numerical directly in the numerical is the numerical directly in the numerical directly in the numerical is the numerical directly in the method only only $\sum_{\Delta} y$ in components are a ∂x method wields ∂y od which $\sum_{\mathbf{a}}$ o as the frequency inc The review \bullet the single-layer of the single-layer, adjoint double-layer, adjoint double-layer, adjoint double- $\sum_{\mathbf{a} \in \mathbb{R}^n} \mathbf{a} \times \mathbf{b}$ operators S $\sum_{\mathbf{b} \in \mathbb{R}^n} \mathbf{b} \times \mathbf{b}$ operators $\sum_{\mathbf{c} \in \mathbb{R}^n} \mathbf{b} \times \mathbf{b}$ operators S $\sum_{\mathbf{c} \in \mathbb{R}^n} \mathbf{b} \times \mathbf{c}$ and H d n d $S \qquad x,y \quad y \, ds \, y$ X, Y (y ds(y

D
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\frac{x_1y}{n\underline{x}} \underline{y} ds \underline{y}
$$
, H $\frac{2}{n\underline{x}} \underline{y} dy$.
a $\frac{1}{2}$ $\frac{2}{2}$ $\frac{2$

$$
\text{for } \mathbf{a} \to \mathbf{a} \text{ on } \mathbf{f}_0 \text{ such that } \mathbf{a} \text{ is a constant.}
$$

 Δ $\sum_{i=1}^{\infty}$ ion is a second-kind integral equation is a second-kind integral equation is a second-kind in the unknown solution determines the unknown solution $\sum_{i=1}^{\infty}$ and unknown solution $\sum_{i=1}^{\infty}$ and unkn $v = \frac{1}{n}$ and $\frac{1}{n}$ a $\frac{1}{n}$ on \sim a form of the $\frac{1}{n}$ is $\frac{1$ is sufficient of integral operators \mathbf{C}^1 is sufficient operators D and S in (1.5) are compact on standard function spaces, so that A is a compact perturbation of the identity operator. $U\cap V$ classical arguments based on this property on the can show that α is property that standard numerical standard numerical standard numerical standard numerical standard numerical standard numerical standard numer chives like $\lim_{\Delta x \to 0} \ln \text{ and co-oc}_{\Delta}$ on \mathcal{L} od in i.e. it o ynomial basis functions lead only $y \triangleleft d$ and $y \triangleleft d$ array values vnumerical solutions vnumerical solutions values of $\sum_{i=1}^{\infty} a_i$ σ^2 to \mathcal{F}

$$
\mathbf{V} - \mathbf{V}_N \qquad \mathbf{C} \quad \inf_{\mathbf{N} \subseteq \mathbf{S_N}} \quad \mathbf{V} - N \quad ,
$$

 \mathbf{S}_N denotes the finite-dimensional approximation space in the disputation distribution space in the disputation of discrete \mathbf{S}_N denotes the disputation space in the disputation of disputation space in the disp cretisation parameters, e.g. the dimension of the space \mathbf{S}_N . More precisely, for properlydind_ia in method and collocation method collocation methods, the methods, th λ_{α} a λ_{α} and λ_{0} is a c such that λ_{0} is λ_{1} . \S for a little more details. B_{Δ} d on (i.17) one can think of the numerical analysis of \circ or secation for scattering \sim

problems as \sim regions as \sim regions as \sim relations:

d not ood kd nd n n d \mathbb{R} on a \mathbb{C} . S_N o λ best approximation error inf N sn $V - N$ is growing as possible as **k** α^c no α y d nd on **k** and so denote them α^c

 $\overline{\text{cof}}$ of $\overline{\text{c}}$ in $\overline{\text{c}}$ in $\overline{\text{c}}$ in $\overline{\text{c}}$ in $\overline{\text{c}}$ on $\overline{\text{c}}$ in $\overline{\text{c}}$ on k, or for $\frac{1}{2}$ again indicate boundedness $\frac{1}{2}$ or $\frac{1}{2}$ $\frac{1}{2}$

$$
\begin{array}{ccccccccc}\n\text{d} & n & \text{of} & \text{ood} & \text{mod} & \text{of} & \text{mod} & \text{in} & \text{in} & \text{in} & \text{in} & \text{in} & \text{in} & \text{out} & \text
$$

For α and a $\sum_{n=1}^{\infty} \alpha$ is the solution of α is the best approximate best approximate between α tion of \bullet in constant for each fixed N as k \bullet constant analysis of the $\sum_{i=1}^{\infty}$ $\sum_{i=1}^{\infty}$ in § \bar{c} cassical error analysis results for second-kind in the cond-kind integral equations that $\frac{1}{2}$ \sum od for all sufficiently all $\sum_{i=1}^n N_i N_i$ (N $\sum_{i=1}^n N_i$ or $\sum_{i=1}^n N_i$ and $\sum_{i=1}^n N_i$ appears the wavenumber $\sum_{i=1}^n N_i$ and $\sum_{i=1$ non-in-gyinside the σ^2 the clear σ^2 operator σ^2 in σ^2 i

information on either on the constant C depends on the parameter k or fixed N, the constant C depends on the parameter k; or \mathbb{R} depends on the parameter k; or \mathbb{R} depends on the parameter k; or \mathbb{R} depends \int_{∞}^{∞} o^t \int_{0}^{∞} o^t for \sim d CN

for θ some positive constants B and θ in the equation θ is uniquely solvable. Moreover if θ is uniquely solvable. Moreover if θ is uniquely solvable. Moreover if θ is uniquely solvable. The equation of $\text{tr}\left(\frac{1}{2}\right)$ in $\text{tr}\left(\frac{1}{2}\right)$ and $\text{tr}\left(\frac{1}{2}\right)$ of $\text{tr}\left(\frac{1}{2}\right)$ in any finite dimensional λ_{α} C S_N, L₂(Γ), i.e. s_N, s_N, c_a $a_{\mathbf{V}_N}$, w_N $f_{\mathbf{V}_N}$ $v_{\mathbf{V}_N}$ $v_{\mathbf{V}_N}$, $f_{\mathbf{V}_N}$, $f_{\mathbf{V}_N}$ then we have the error estimate (1.7) with C B/α. Therefore one potential way to answer

operators such a point and \sim on $n \geq x$ or e $n = a \cdot 20 \cdot d$ all rules. The partition of unity is designed to localise around to localise around special points ζ coo \sum_{Δ} on on x n_{Δ} y iii namely x ζ and $\tan \frac{1}{2}y$ on arccos and arccos arccos arccos (2.4) vanishes; (iii) shadow boundary points $n_x \cdot a$ (As n_{n-1} on one costs $a_n \cdot a_n$ or a_n or a_n points. These points is is a high-free $\mathbf{u} \sim \mathbf{u}$ of $\mathbf{u} \sim \mathbf{v}$. Ny $\mathbf{v} \sim \mathbf{u}$, od \mathbf{v} is method is me not based on a Galerikin formulation, the analysis of its k-robustness is a challenging open in a challenging o problem. We shall return to methods for october λ integrals for output λ

oyon no a \mathbb{R} y cn o $n = \sqrt[n]{a}$, \mathbb{R} $n = \sqrt[n]{a}$ $n = \sqrt[n]{a}$ out which suggests that the algorithms α the algorithms for the algorithms for the superior α in the combined to compute \mathbb{R} from \mathbb{R} in yerming convex polygons.

 I in the subsection we assume that I is a I is a I in the I is a contract of I is a contract of I in I is a contract of I is a contract of I in I is a cont $\int \ln a \sin \theta$ is not to the two tangency points $\ln a$ and $\ln a$ and $\ln a$ and $\ln a$ illuminated $\ln a$ is $\ln a$ and $\ln a$ into an illuminated zone. and a shadow zone is a depicted in Figure 1. Letting γ is a γ is codic $\frac{1}{2}$ a $\frac{1}{2}$ of $\frac{1}{2}$ of $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ for $\frac{1}{2}$ $\frac{1}{2}$ i $1, \ldots, 0$ metric $\alpha^n \alpha^n \alpha^n$ may then be written α^n $v_{\mathcal{S}}$, k for convenience

 $v_{s,k}$ k exp(ik) a V s,k .

 \mathbf{n} = $\mathbf{s}_1 \cdot \mathbf{s}_2$, $\mathbf{s}_3 \cdot \mathbf{s}_4$ is covered by four \mathbf{s}_1 , \mathbf{s}_2 , \mathbf{s}_3 , \mathbf{s}_4 , \mathbf{s}_5 and \mathbf{M}_2 in it is \mathbf{M}_4 being small neighbourhood of the tangency points t₁ and t₂, and \mathbf{M}_3 and \mathbf{M}_4

Considering the numerical analysis of the \mathbf{m} oder \mathbf{c}_Δ and \mathbf{c}_Δ of two key \mathbf{m} and α din § α and α need α for α of V ϵ , k ith respect to see the intervention of $\lim_{n \to \infty} \frac{1}{n}$ in the intervention $\lim_{n \to \infty} \frac{1}{n}$ in the intervention $\lim_{n \to \infty} \frac{1}{n}$ intervention $\lim_{n \to \infty} \frac{1}{n}$ ndeed $\sum_{i=1}^{\infty}$ for $\sum_{i=1}^{\infty}$ on night decay of **V** in the decay decay $\sum_{i=1}^{\infty}$ requires $\sum_{i=1}^{\infty}$ a substantial study of the theory of the theory of the α \mathbf{g} for \mathbf{g} the α f oo na result is presented.

For all L,M $\mathbb N$ $\{$ }, the function V \mathcal{S}, k admits a decomposition of the

form:

$$
V_{\mathcal{S},k} \left[\begin{array}{cccc} L^M & & \\ & k^{-1} \tfrac{3-2}{2} & \tfrac{3}{2} & \tfrac{3}{2} \\ & & \tfrac{3}{2} & \tfrac{3}{2} \end{array}\right] \quad R_{LM} \mathcal{S},k \quad ,
$$

for s \blacksquare , , where the remainder term has its nth derivative bounded, for n $\mathbb N$ $\{$ $\}$, by

$$
|\mathbf{D}^n\mathbf{R}_{LM}\mathbf{S}_l\mathbf{k}| \mathbf{C}_{LMn}\mathbf{R}^{n} \mathbf{R}^{n-3},
$$

where μ = $\frac{2}{3}$, μ and $C_{L,M,n}$ is independent of k. The functions b and Z are C \qquad -periodic functions. Z has simple zeros at ${\sf t}_1$ and ${\sf t}_2$

decrease exponentially but in a very oscillating way. The asymptotics in (2.10) may be deduced by applying the theory of residues to the contour integral defining Ψ - see [8, p.393], [18, Lemma 8]. More details are in [30]. Combining these asymptotics with Theorem 2.1, the following estimates for the derivatives of V are proved in [30].

For all n $\mathbb N$ $\{$ } there exist constants C_n > independent of k and s for all k sufficiently large,

$$
|D^{n}V_{s}S_{L}k| C_{n} \begin{cases} \n\frac{1}{k-1} + \frac{1}{k-1} - \frac{1}{k-1} \\ \n\frac{1}{k-1} + \frac{1}{k-1} - \frac{1}{k-1} \n\end{cases}
$$

where $s = s - t_1, t_2 - s$. These estimates are uniform in s \ldots

This statement follows from \mathbf{S}_1 but is in a somewhat simpler form \mathbf{S}_2 and \mathbf{S}_3 \blacktriangleright n is the estential point which follows from the interval point is that for s in the interval point \mathbf{q} is that for s in the interval point \mathbf{q} is the interval point \mathbf{q} is the interval point \mathbf{q} and ond d a $y^r o \frac{1}{2!}$, t_2 and ne | s

that by choosing p to grow slightly faster than k ¹ ⁹ we preserve the accuracy of the method as k increases. Numerical results in [30] support this result. Before leaving this discussion we mention that using the asymptotics (2.10) when s is near to but less than t¹ (i.e. in the shadow region but near the transition point), then the first term in (2.7) has the asymptotics (as k → ∞) : k −1 3 b⁰ ⁰ exp(ik|Z(s | 3 /3) exp(i (¹ k 1 3 |Z(s |) exp(−ℑ(¹ k 1 3 |Z(s | . (2.15) Since ¹ is in the first quadrant of the complex plane (see (2.10)), (2.15) contains two oscillatory factors, one oscillating with scale k and one with scale k , damped by the exponentially 1 3 decaying third term. These two scales were modelled in the basis functions used in the collo-

 $c_{\hat{a}}$ on \bullet od of \hat{c} and \hat{c} oo \hat{c} no account \bullet \hat{c} and \bullet \bullet \hat{c} into \bullet \hat{c} for $\operatorname{ad} \circ A$ on $\operatorname{ad} \circ X$.

no no a nonocating bodies.

 $\ln \xi$ might $\ln \xi$ we have $\sin \xi$ of $\cos \theta$ high $\sin \theta$ frequency boundary $\sin \theta$ and $\sin \theta$ research on the relations. We turn in the relations of the section to the section to the section to the second to the second to the second to the section to the section to the section to the second to the second to the se ond of λ for $n_A \blacktriangleleft y$ the problem of the problem of the stability constant C in (1.7). We note that, while the emphasis of the emphasi tion \bullet oder specifically adapted to the results of th \sim and \log is stability analysis and conditioning for conventional piecewise polynomial piec $\text{o} \text{nd}_{\Delta} \text{y}$ $\text{on} \text{d}_{\Delta}$ od at f^{A} ncy \mathbb{R} noted a idy a in $c_{\mathbf{a}}$ and \mathbb{R} is coefficient and ond on the stability constant C in the case when v_N is defined by the Galery of $\sum_{i=1}^{\infty}$ or $\sum_{i=1}^{\infty}$ \mathbf{y} is a set of \mathbf{y} C B \mathbb{R}

B and a continuity and coeff y constants in \mathbf{d} constants are constants ar a do norms of A and its inverse and \mathbf{L}^2 and dyear or \mathbf{L}^2 For V, W in L^2 and with k d norm of a bounded many \mathbb{R}^4 a bounded in a bounded linear operator. on L^2 ,

$$
|a_{\mathbf{V},\mathbf{W}}| \, |\, A_{\mathbf{V},\mathbf{W}}|_{L^2(\cdot)}| \, A_{\mathbf{V}}|_{L^2(\cdot)} \, w_{L^2(\cdot)}|_{L^2(\cdot)} \, w_{L^2(\cdot)}.
$$

 A is a possible value for the constant B in A in A is A in A is A in A is A w A v in the above integration of \sum_{Δ} is the smallest possible value for the smallest po which ϕ and ϕ in the second of the second of the second of the inequality ϕ .

$$
\mathbf{A}\mathbf{v}_{L^2(\cdot)}\mathbf{v}_{L^2(\cdot)}\left|\mathbf{A}\mathbf{v},\mathbf{v}_{L^2(\cdot)}\right|\left|\mathbf{a}\mathbf{v},\mathbf{v}\right|\left|\mathbf{v}\right|_{L^2(\cdot)}^2,
$$

 $\overline{0}$ λ

$$
A^{-1} \qquad \qquad ^{-1}.
$$

 T_A is **B/** is bounded by **condition number of the operator A**

$$
\begin{array}{cccc}\n\mathbf{B} & \text{cond } \mathbf{A} & \mathbf{A} & \mathbf{A}^{-1}\n\end{array}
$$

 T on $\bigoplus_{\alpha=0}^{\infty}$ on \bullet or dying condition number of A and its dependence on k, which will be a main topic of the section. Another motivation is the following. The following $\sum_{i=1}^{\infty}$ and $\sum_{i=1}^{\infty}$ and $\sum_{i=1}^{\infty}$ and $\sum_{i=1}^{\infty}$ and $\sum_{i=1}^{\infty}$ and $\sum_{i=1}^{\infty}$ and $\sum_{i=1$ in \mathbb{R}^N is \mathbb{R}^N is only useful if a is coefficient which we will see below is known to be the case if $\sum_{i=1}^n a_i c_i$ coercives on $\sum_{i=1}^n a_i$ is $\sum_{i=1}^n a_i$ is a isomorphism or not a is coefficient or not a is coefficient of $\sum_{i=1}^n a_i$ is a isomorphism or not a is coefficient of $\sum_{i=1}^n a_i$ is a isomorphism o

 $\mathrm{d}_{\mathbf{r}}$

 $\sim N$ \sim k

$$
A \underbrace{A^n \underbrace{\pi_0}_{c \text{ on } a} \underbrace{a^n}_{c \text{ on } a} \underbrace{\pi_1}_{c \text{ on } a} \underbrace{c \underbrace{a}_{\text{ on } a} \underbrace{a}_{\text{ on }
$$

$$
\mathsf{R}_0^{-1} \quad \mathsf{k}, \quad \mathsf{R}_1 \quad \mathsf{R} \quad \mathsf{
$$

which satisfies (3.21) with l¹ l²

 10 mm Bonner, 12.9 cm , $12.9 \text{$ diffraction coefficients in $A \rightarrow \infty$ and C_4 acoustic wave scattering, SIAM J. Numer. Anal. 43, 1202–1230. $1-\mathbf{B}$ rakhange, H. and \mathbf{a} , M. and \mathbf{b} , P. and \mathbf{c} , \mathbf{d} , H_{NQ} zcennung Schwingungsgleichungsgleichungsgleichungsgleichungsgleichungsgleichung, α B no A A Mon o. A., C is C C i with $\sim d$ computational times for scattering problems σ arbitrarily high frequency: $\text{co}_{\mathbb{R}}^{\mathbb{Z}}$ c_{as} Phil. Trans. R. Soc. Lond. A. $\text{d}_{\mathbf{a}}$ B no and A A and A and A are A . An O dimensional surface $\lambda c_{\rm A}$ in a λ ₇₆. J. Comp. Appl. Math. 204 λ 64 [14] Bruno, O.P., Geuzaine, C.A., Reitich, F. (2005). On the O(1) solution of multiple- \mathfrak{c}_{Δ} in $\alpha \lambda_{\mathcal{B}}$ IEEE Trans. Magn. $\mathfrak{p}_{\mathbf{e}}$ α 15 no and 15 and 1 erated in order **Soc. R. Soc. Lond.** A 457, 293, 293, 2921-2934. α \mathbf{B} no, \mathbf{A} communication. 17 Buffa, A. and Sauter, S. (2006). On the acoustic single layer potential: Stabilisation and σ is σ an alysis. SIAM J. Sci. Comput. σ is σ determined the σ \mathbb{R} Bush \mathbb{R} bushes as \mathbb{R} short-wave asymptotic behaviour in the problem by smooth contribution of the Russian of Trudy Mat. Inst. Steklov. \bullet The Abbreviated d $E = \frac{1}{2}$ in the summary in the shortwave asymptotic limit in the problem of different problem of different $\frac{1}{2}$ $\overline{\text{co}}$ \bullet $\overline{\text{od}}$ $\overline{\text{co}}$ Soviet Physics Doklady $\overline{\text{kg}}$, $\overline{\text{kg}}$, $\overline{\text{d}}$ [19] Buslaev, V.S. (1975). The asymptotic behavior of the spectral characteristics of exterior problems for the odinger operator (in Russian). Izv. Akad. Nauk SSSR Ser. Mat. α ₉ and α ₂ anglic on α ₂. **Math. USSR–Izv.** α \bullet and id N., \bullet s. N., i. Langdon, S., and Lindner, M. (2007). Condition ntegral boundary integral boundary integral potential boundary integral operators in a count of c_{Δ} the sac Newton Institute \mathbb{R}^n of \mathbb{R}^n and $\$ N on n i \bullet and id N_{raham}, Landon, And Lindner, **C**ondi for number \mathcal{L} for compies \mathcal{L} integral operators in a compies operators in a couple in a c onday \Box dic promin preparation. \bullet and id N \bullet and and Ritter, A is \bullet number number number of \bullet number number number number number number number number number of \bullet number number number number number of \bullet number number of \bullet number of ond_ay \mathbf{e}_n od for an acoustic scattering problem. **R. Soc. Lond.** A. 362, 647–671.

 L^{\star} de \mathbb{C}^{c} conand L^{\star} den L and L^{\star} α is α must be a multiplet of α . M n M and Landon, \therefore An hp-B M for $f^* \rightarrow$ ncy c_{Δ} in y $co**r**$ ² oyon ocdings of A \overline{M} y of \overline{M} M o B N M $\frac{1}{\sqrt{3}}$ N a a c in and cocdic o

