High frequency scattering by convex curvilinear polygons

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Abstract

We consider scattering of a time-harmonic acoustic incident plane wave by a sound

$$u(\mathrm{x}) = 0, \quad \mathrm{x} \in \Gamma, \tag{2}$$

together with the Sommerfeld radiation condition

$$\lim_{r \to \infty} r^{1/2} \left(\frac{\partial u^s}{\partial r} - \mathrm{i}ku^s \right) = 0, \tag{3}$$

on the scattered field $u^s := u - u^i$, where $r := |\mathrm{x}|$ and the limit holds uniformly in all directions $\mathrm{x}/|\mathrm{x}|$. Existence and uniqueness of a solution $u \in C^2(D) \cap H^1_{\mathrm{loc}}(D)$ to (1)–(3) is well known - see [4, §2] for a full discussion. Using Green's theorem we have the representation [5, theorem 3.12]

$$u(\mathrm{x}) = u^i(\mathrm{x}) - \int_\Gamma \Phi(\mathrm{x}, \mathrm{y}) \frac{\partial u}{\partial \mathrm{n}}(\mathrm{y})\, \mathrm{d}s(\mathrm{y}), \quad \mathrm{x} \in D.$$

Here $\Phi(\mathrm{x}, \mathrm{y}) := (\mathrm{i}/4) H_0^{(1)}(k|\mathrm{x} - \mathrm{y}|)$ is the fundamental solution of the two-dimensional Helmholtz equation and $\partial/\partial\mathrm{n}$ represents the derivative with respect to the unit outward normal vector n.

Knowledge of the complementary boundary data $\partial u/\partial\mathrm{n} \in L^2(\Gamma)$ thus gives an expression for the total field at any point. Following the usual coupling procedure, we obtain the well known second kind boundary integral equation

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(the points on the boundary where $\mathbf{n}.\mathbf{d} = 0$),

$$\frac{1}{k}\frac{\partial u}{\partial \mathbf{n}} \to \Psi := \begin{cases} \frac{2}{k}\frac{\partial u^i}{\partial \mathbf{n}} & \text{in the illuminated region, for which } \mathbf{n}.\mathbf{d} < 0, \\ 0 & \text{in the shadow region, for which } \mathbf{n}.\mathbf{d} > 0. \end{cases}$$

Thus writing

$$\frac{1}{k}\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = e^{ik\mathbf{x}.\mathbf{d}} w(\mathbf{x}), \quad \mathbf{x} \quad \Gamma, \tag{5}$$

in (4) leads to a second kind boundary integral equation for a new unknown function w, which is more amenable to approximation by piecewise polynomials for large k than $\partial u/\partial \mathbf{n}$, since it approaches a constant in the illuminated and shadow regions (away from corners and shadow boundaries) as $k \to \infty$.

This approach was first attempted in [1], for problems of scattering by smooth convex obstacles, with numerical results and analysis suggesting that the number of degrees of freedom required to maintain accuracy need only grow with order $k^{1/3}$ as k increases (compared to order k for standard schemes). Combining this approach with a mesh refinement, concentrating the degrees of freedom near the shadow boundary, it appears that the order $k^{1/3}$ requirement can be removed altogether. A rigorous analysis in [6] demonstrates that increasing the number of degrees of freedom with order $k^{1/9}$ is su□cient to maintain accuracy, and numerical results in [2,6,7] (the latter with the advantage of a sparse linear system) suggest that a prescribed level of accuracy can be achieved with a number of degrees of freedom that is independent of k.

The schemes of [1,2,6,7] all assume smooth Γ and perform poorly if Γ has corners, since in this case the oscillatory behaviour of the field di□racted by the corners is not well represented by the function Ψ. The simplest obstacle with corners, a straight-sided convex254(coo1(y)-247(o)1(bs)-1(t)1(a-)-254(ssugg)5811(sided)-254(c smooth functions $v_\pm$. These functions decay sufficiently quickly that the number of degrees of freedom required to maintain the accuracy of their best L^2 approximation from a space of piecewise polynomials supported on a specific graded mesh, with a higher concentration of mesh points closer to the corners of the polygon, grows only logarithmically with respect to k as $k \to \infty$. This appears to be the best rigorous numerical analysis result to date for a problem of scattering by bounded obstacles.

In this paper, we consider the case where the scatterer has curved sides, meeting at corners. In this case, using the formulation (6) directly would not be appropriate, as the terms $v_\pm$ would still oscillate. Instead, a slightly different approach is required, combining the ideas of (5) and (6). We now write the unknown function $\partial u/\partial \mathrm{n}$ as

$$\frac{1}{k}\frac{\partial u}{\partial \mathrm{n}}(\mathrm{x}(s)) = \mathrm{e}^{\mathrm{i}k\mathbf{x}(s).\mathbf{d}} w(s) + \mathrm{e}^{\mathrm{i}ks} v_+(s) + \mathrm{e}^{-\mathrm{i}ks} v_-(s), \quad s \in [0, L], \tag{7}$$

where again $\mathrm{x}(s)$ denotes arc-length parametrisation on Γ, and now the functions w and $v_\pm$ must each be determined. Our rationale behind this representation is that the oscillatory behaviour of the "reflected field" (the scattered field in the absence of diffraction) will be well represented by $\mathrm{e}^{\mathrm{i}k\mathbf{x}(s).\mathbf{d}}$, and the oscillatory behaviour of the "diffracted field" travelling along each side of the obstacle away from the corners will be well represented by $\mathrm{e}^{\pm \mathrm{i}ks}$. We know, from results in [1,2,6,7], that in the absence of corners the representation (7) will work well, with $v_\pm = 0$ and w slowly oscillating away from shadow boundaries. Further, from results in [4] we know that if the polygon has straight sides then the representation (7) again works well, with $v_\pm$ and $w(s) = \mathrm{e}^{-\mathrm{i}k\mathbf{x}(s).\mathbf{d}}\Psi(\mathrm{x}(s))$ all non oscillatory, and $v_\pm$ highly peaked near the corners and rapidly decaying away from the corners.

In the next section we describe our Galerkin boundary element method. The approximation space we use consists of the products of plane waves $\mathrm{e}^{\mathrm{i}k\mathbf{x}(s).\mathbf{d}}$ with piecewise polynomials supported on a uniform mesh on the illuminated sides (to approximate w in (7)), together with the products of plane waves $\mathrm{e}^{\pm \mathrm{i}ks}$ with piecewise polynomials supported on graded meshes on each side of the polygon, with these meshes graded towards the corners (to approximate $v_\pm$ in (7)). In §3 we demonstrate via numerical experiments that this approach only appears to require a logarithmic increase in the number of degrees of freedom, with respect to k, in order to maintain accuracy as k increases. Finally in §4 we present some conclusions.

For simplicity we assume that $\mathrm{n}.\mathrm{d} = 0$, i.e. we assume that the "shadow boundary" between the illuminated and shadow sides occurs at a corner, with no grazing incidence. If this were not the case, special care would be needed in the "transition zone" around the shadow boundary $\mathrm{n}.\mathrm{d} = 0$ (see e.g. [2,6]).

2 The Galerkin boundary element method

We begin by defining some notation, as in Figure 1. We write the boundary

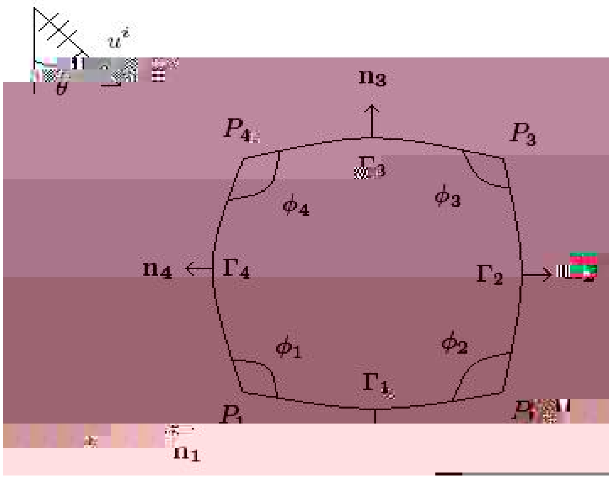

Fig. 1. Scattering by a curvilinear polygon

$\lambda := 2\pi/k$, we begin by defining a graded mesh on a segment $[0, A]$, for $A > \lambda$. This is the same graded mesh as that used in [4] for the case that each Γ_j is a straight line. We use a composite mesh, with a polynomial grading on $[0, \lambda]$, with the N points accumulating near the origin, and a geometric grading on $[\lambda, A]$, with the $\hat{N}_{A,\lambda,q}$ points becoming more widely spaced away from λ, as shown in figure 2. For large N, $\hat{N}_{A,\lambda,q}$ is proportional to N.

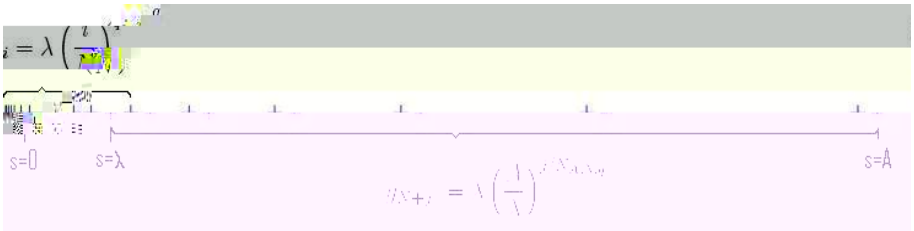

Fig. 2. Composite mesh on $[0, A]$

Definition 1 *For* $A > \lambda > 0$, $q > 0$, $N = 2, 3, \ldots$, *the mesh* $\Lambda_{N,A,\lambda,q}$,Ff3yFfNFfFfNffniq./la

$$\Pi_{\Gamma_j^+,\nu} := \{\sigma \quad L^2(0,L) : \sigma|_{(\tilde{L}_{j-1}+y_{m-1},\tilde{L}_{j-1}+y_m)} \text{ is a polynomial of degree } \le \nu,$$
$$\text{for } m = 1,\dots,N+\hat{N}_{L_j,\lambda,q_j}, \text{ and } \sigma|_{(0,\tilde{L}_{j-1}) \quad (\tilde{L}_j,L)} = 0\},$$
$$\Pi_{\Gamma_j^-,\nu} := \{\sigma \quad L^2(0,L) : \sigma|_{(\tilde{L}_j-\tilde{y}_m,\tilde{L}_j-\tilde{y}_{m-1})} \text{ is a polynomial of degree } \le \nu,$$
$$\text{for } m = 1,\dots,N+\hat{N}_{L_j,\lambda,q_{j+1}}, \text{ and } \sigma|_{(0,\tilde{L}_{j-1}) \quad (\tilde{L}_j,L)} = 0\},$$

where $\{y_0,\dots,y_{N+\hat{N}_{L_j,\lambda,q_j}}\}$ and $\{\tilde{y}_0,\dots,\tilde{y}_{N+\hat{N}_{L_j,\lambda,q_{j+1}}}\}$ denote the points of the meshes $\Lambda_{N,L_j,\lambda,q_j}$ and $\Lambda_{N,L_j,\lambda,q_{j+1}}$ respectively.

Finally we define a uniform mesh on each illuminated side. For $j = n_s + 1,\dots,n$, the mesh $\Gamma_j^u := \{z_0,\dots,z_{N_j^u}\}$ consists of the points

$$z_i = \tilde{L}_{j-1} + \frac{i}{N_j^u} L_j, \quad i = 0,\dots,N_j^u. \qquad L \qquad L$$

where ρ_j is the jth basis function and M_N is the dimension of $V_{N,0}$. For $p = 1, \ldots, n$, we define $n_p^{\pm}$ to be the number of points of $\Gamma_p^{\pm}$, so $n_p^{+} := N + \hat{N}_{L_p,\lambda,q_p}$,
n

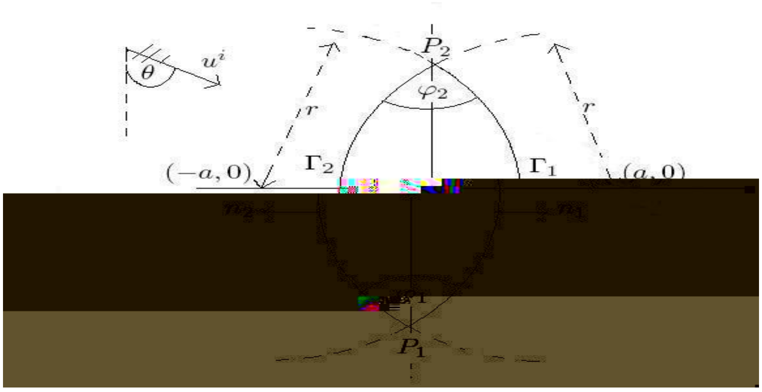

Fig. 3. Scattering by a two sided curvilinear polygon

left of the x_2-axis. The internal angles at each corner are $\phi_i = 2\cos^{-1}(a/r)$, $i = 1, 2$, and an arc-length parametrisation of Γ is

$$\mathrm{x}(s) = \begin{cases} \left(-a + r\cos(\frac{s}{r} - \cos^{-1}\frac{a}{r}), r\sin(\frac{s}{r} - \cos^{-1}\frac{a}{r})\right), & s \in [0, 2r\cos^{-1}\frac{a}{r}), \\ \left(a + r\cos(\frac{s}{r} - 3\cos^{-1}\frac{a}{r} + \pi), r\sin(\frac{s}{r} - 3\cos^{-1}\frac{a}{r} + \pi)\right), & s \in [2r\cos^{-1}\frac{a}{r}, 4r\cos^{-1}\frac{a}{r}). \end{cases}$$

We choose $\theta = \pi/2$, $r = 3$ and $a = 1.5$, so that each side of the polygon is of length 2π and the obstacle has boundary length 4π. In our experiments we take $N_j^u = N$, $\nu = 0$, so that we are approximating by piecewise constants multiplied by plane wave basis functions on the overlapping meshes, and $\eta = -k$, this choice motivated by a desire to minimise the condition number of the resulting linear system (see [3] and the references therein for details).

In Figure 4 we plot $|\varphi$ 8 > n iur> pl >

as N increases. For small values of N the effect of multiplying plane wave basis

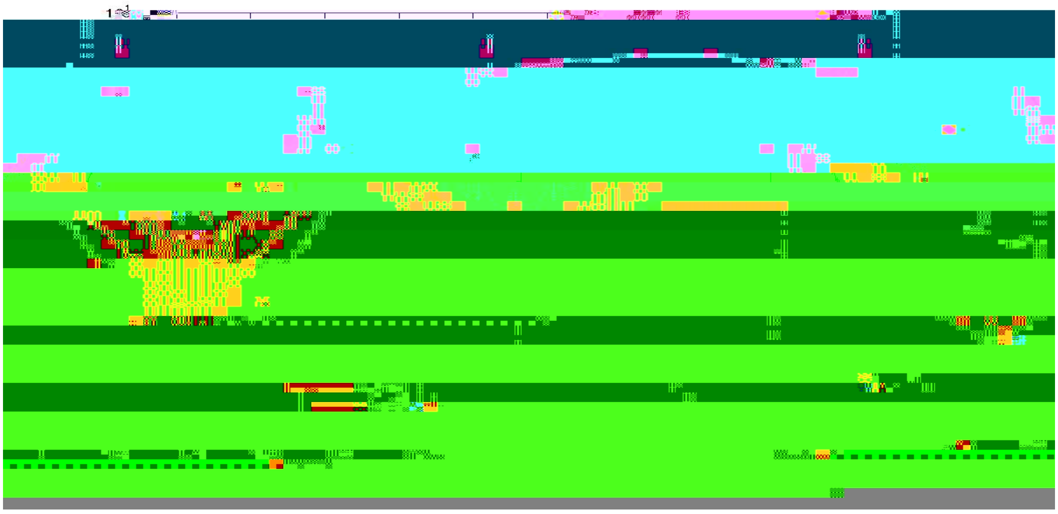

Fig. 5. Comparison of solutions for various k, each computed with $N = 128$.

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