THE PML FOR ROUGH SURFACE SCATTERING

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Abstract. In this paper we investigate the use of the perfectly matched layer (PML) to truncate a time harmonic rough surface scattering problem in the direction away from the scatterer. We prove existence and uniqueness of the solution of the truncated problem as well as an error estimate depending on the thickness and composition of the layer. This global error estimate predicts a linear rate of convergence (under some conditions on the relative size of the real and imaginary parts of the PML function) rather than the usual exponential rate. We then consider scattering by a half-space and show that the solution of the PML truncated problem converges globally at most quadratically (up to logarithmic factors), providing support for our general theory. However we also prove exponential convergence onvRough surface scattering problems are the subect of intensive studies in the er

a view to developing both rigorous methods of computation and approximate, asymptotic, or statistical methods (see e.g. the reviews and monographs by Ogilvy [19], Voronovich [22], Saillard Sentenac [20], Warnick Chew [23], and DeSanto [10]). The standard way of approximating such problems is to use boundary integral techniques. However, variational domain formulations discretised with nite elements are also widely used, especially in the case when the boundary is periodic (e.g. [3, 12]). Moreover, using variational techniques [4], we have been able to extend the existence and uniqueness theory for the sound soft acoustic scattering problem to more general surfaces than was possible using integral equation techniques.

equipped with the parameter dependent inner-product

$$(u,v)_{V_a} := \int_{S_a} (\nabla u \cdot \nabla \bar{v} + k^2 u \bar{v})\, dx$$

where $\bar{v}$ denotes the complex conjugate of v, and $k > 0$ is the wave-number of the acoustic field (see equation (1.3) below). The resulting norm is denoted $\|v\|_{V_a} := \sqrt{(v,v)_{V_a}}$.

Now we can state the time harmonic scattering problem we shall study. Given $g \in L^2(D)$, with support in S_H for some $H \geq f_+$, and the wavenumber $k > 0$, we wish to approximate the acoustic field $u \in V_a$ for every $a \geq f_+$ that satisfies the Helmholtz equation

$$\Delta u + k^2 u = g$$

This estimate can then be used to prove existence, uniqueness and convergence for a general rough surface satisfying (1.8). In particular, under appropriate conditions, we prove a first order convergence rate for the error in the global V_H norm as the layer thickness increases (instead of the more standard exponential rate for bounded scatterers). By considering in detail scattering from a flat surface, we show that, in this norm, convergence cannot be faster than quadratic (up to logarithmic factors) due to the exponential damping of the PML solution along the surface. We also show that, for a half-space scatterer, exponential convergence is observed on compact subsets.

The analysis we shall give is just a first step. We shall use a PML only to truncate in the direction vertically away from the rough surface (for the use of an alternative method for truncation, the pole condition, and its relationship to (1.4) in the 2D case see [2]). A practical calculation also requires truncation laterally. This is true also for boundary integral equation approaches and is a well studied problem in that case (see for example [6]). We do not estimate the error from this truncation on the variational PML method. In addition we do not estimate the error in the resulting finite element scheme.

Our PML convergence proof suggests that, in the worst case, the method may converge slowly as the virtual thickness of the PML layer expands, and, in addition, it can be difficult to determine the optimal PML parameters in realistic simulations. We are thus motivated to combine the variational PML method with an iterative improvement scheme motivated by the work of Liu and Jin [18] using an integral operator with a smooth kernel to provide a correction to the variational PML scheme. We prove that this iterative approach converges. The cost of each iteration includes the cost of evaluating the integral operator which can be done rapidly using the Fast Fourier Transform since the integral operator in this case has a smooth kernel, and the solution of a finite element problem on a truncated portion of the strip S_H.

The outline of the paper is as follows. In the next section we recall the variational formulation of the rough surface problem in [4] and provide a variational formulation using the PML via a change of variables approach. Then in Section 3 we prove a general error estimate for the solution of the truncated PML problem using the variational formulation and a Fourier analysis of the PML layer. This result proves only first order convergence in the global V_H norm. The sharpness of this convergence result is then investigated in Section 4 where we derive some estimates for the PML solution when the scatterer is flat. On the one hand, these estimates show that, even in this simple case, the PML solution does not converge exponentially rapidly to the exact solution in the global V_H norm, since the PML solution does not have the right asymptotic behaviour at infinity. On the other hand, we show that on compact subsets of S_H the PML solution does converge exponentially rapidly as the layer thickness increases. Our analysis of the flat scatterer, where we are able to obtain exact representations of the error in the PML approximation, is somewhat reminiscent of recent analysis of the time domain PML for simple geometries in [11]. In Section 5 we show how an iterative scheme to improve the PML solution can be constructed and prove its convergence using the estimates from Section 3. Finally, we present some very limited numerical examples testing our theory in Section 6 and finally draw some conclusions in Section 7. We shall present the method and our analysis in $\mathbb{R}^n$, $n = 2, 3$, but the numerical results are in $\mathbb{R}^2$.

2. Variational Formulation and the PML.

We start by recalling a variational formulation for the sound soft rough surface scattering problem used in [4]. In order to write down this variational formulation, we first define the appropriate Dirichlet-to-Neumann map $T : H^{1/2}(\Gamma_H) \to H^{-1/2}(\Gamma_H)$ for the domain U_H. Precisely, for a given function $\varphi \in H^{1/2}(\Gamma_H)$, we have

$$T = F^{-1} M_z F \tag{2.1}$$

where F is the Fourier transform operator defined in (1.5) and M_z is the operator in transform space of multiplication by $z(\xi)$ given by

$$z(\xi) =$$

Using this boundary condition and standard variational arguments (see [4] for details) we can pose (1.3), (1.4) and (1.6) as the variational problem of finding the function u V_H such that

$$b(u, \varphi) = -(g, \varphi) \text{ for all } \varphi \quad V_H, \tag{2.3}$$

where the sesquilinear form $b(.,.)$ is given by

$$b(u, \varphi) = \int_{S_H} u \cdot$$

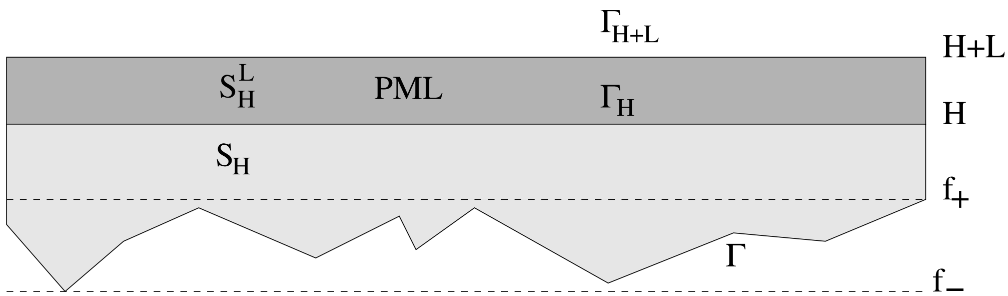

Fig. 2.1. *A schematic showing some of the notation for the PML terminated acoustic rough surface problem. The lightly shaded region S_H is the main computational domain where the Helmholtz equation is satisfied. The domain S_H^L is occupied by the PML. A non-homogeneous Dirichlet boundary condition is applied on Γ and a homogeneous Neumann condition is applied on Γ_{H+L}. The boundary Γ_H is used in computing the Dirichlet to Neumann maps.*

In general if the quantity $(k\tilde{L})$ is large the PML will absorb evanescent modes well, whereas if $(k\tilde{L})$ is large the PML will absorb waves propagating into the PML well.

Formally, the change of variables technique is to require that the solution in the PML, denoted u_p, satisfy the Helmholtz equation in stretched coordinates so

$$\sum_{j=1}^{n-1}\frac{\partial^2 u_p}{\partial x_j^2}+\frac{\partial^2 u_p}{\partial \hat{x}_n^2}+k^2u_p=0 \text{ in } S_H^L.$$

Then changing variables back to standard real coordinates, using the fact that $d\hat{x}_n/dx_n=\sigma$ and the fact that σ depends only on x_n, we obtain the PML equation

$$\sum_{j=1}^{n-1}\frac{\partial}{\partial x_j}\left(\sigma\frac{\partial}{\partial x_j}u_p\right)+\frac{\partial}{\partial x_n}\left(\frac{1}{\sigma}\frac{\partial}{\partial x_n}u_p\right)+k^2\sigma u_p=0 \text{ in } S_H^L.$$

For convenience we then define the appropriate differential operator for the PML,

$$\Delta_p v=\sum_{j=1}^{n-1}\frac{\partial}{\partial x_j}\left(\sigma\frac{\partial}{\partial x_j}v\right)+\frac{\partial}{\partial x_n}\left(\frac{1}{\sigma}\frac{\partial}{\partial x_n}v\right).$$

Of course $\Delta_p=\Delta$ outside the PML.

We can now state the truncated PML problem: we seek $u_p \in V_{H+L}$ such that

$$\Delta_p u_p+k^2\sigma u_p=g \text{ in } S_{H+L}, \tag{2.9}$$

$$\frac{\partial}{\partial x_n}u_p=0 \text{ on } \Gamma_{H+L}, \tag{2.10}$$

where (2.9) is understood in the sense of distributions and (2.10) by duality. The fact that $u_p \in V_{H+L}$ implies that $u_p=0$ on Γ. The choice of a Neumann condition on Γ_{H+L} is essentially arbitrary and we shall compare this choice to the more standard choice of a Dirichlet boundary condition in Section 4.

For the purpose of analysis (for computational purposes the finite element grid covers all S_{H+L} including the PML zone S_H^L as well as S_H) we follow [7] and eliminate the PML by using the Dirichlet to Neumann map for the PML domain above Γ_H. For later use we consider a more general problem than is needed at this stage having non-homogeneous boundary data on Γ_{H+L}. In particular, given $s \in H^{-1/2}(\Gamma_{H+L})$ and $q \in H^{1/2}(\Gamma_H)$ we wish to compute the Dirichlet data on Γ_H for the problem of finding $v \in H^1(S_H^L)$ such that

$$v=q \text{ on } \Gamma_H,$$
$$\Delta_p v+k^2\sigma v=0 \text{ on } S_H^L,$$
$$\frac{1}{\sigma}\ \partial v$$

Since the solution in the PML zone S_H^L

where M_{z_p} is the operator of multiplication by the function z_p defined by

$$z_p(\xi) = z(\xi) \left(\frac{\exp(z(\xi)\tilde{L}) - \exp(-z(\xi)\tilde{L})}{\exp(z(\xi)\tilde{L}) + \exp(-z(\xi)\tilde{L})} \right), \quad \xi$$

where the last inequality follows from the trace theorem that

$$\varphi_{H^{1/2}(\Gamma_H)} \le \sqrt{2}\, \varphi_{V_H} \tag{3.1}$$

for any φ V_H (proved in [4]). Standard operator perturbation theory [14] tells us that B_p is invertible provided $\delta B_{L(V_H}$

where (defining $p = \sqrt{1 - t^2}$)

$$S_1 = \sup_{0 \le t \le 1} 2\frac{\sqrt{1 - t^2}}{\sqrt{1 + t^2}} \quad \exp(-$$

from which the estimate follows. □

Using the argument preceding this theorem, together with the results of the theorem, we have now proved that provided $C_U(\alpha,\beta)$ is sufficiently small, the PML truncated variational problem has a unique solution.

To provide an error estimate we use the inf-sup condition. Since

$$b_p(u,\varphi) = b(u,\varphi) + \int_{\Gamma_H} \overline{\varphi}(T_p - T)u\,ds$$

we have, for any $u \in V_H$, using the trace estimate (3.1), that

$$\sup_{v\in V_H} \frac{|b_p(u,v)|}{\|v\|_{V_H}} \geq \gamma\|u\|_{V_H} - 2\|T - T_p\|_{L(H^{1/2}(\Gamma_H),H^{-1/2}(\Gamma_H))}\|u\|_{V_H}.$$
$$\geq (\gamma - 2C_U(\alpha,\beta))\|u\|_{V_H}$$

From Theorem 3.1 we thus have

as $\alpha \to \infty$, which is much slower than the exponential rate of convergence proved in [16, 15, 7]. We shall, however, show in Section 4 that, even in the simplest case when the rough surface is flat, exponential convergence in the $\|\cdot\|_{V_H}$ norm is not achieved. In fact we shall see in Remark 4.5 that the estimate (3.4) is fairly sharp.

4. A special case: a flat surface. In this section we shall analyze the special case of scattering by a flat surface in $\mathbb{R}^2$ so that for this section $D = U_0 \subset \mathbb{R}^2$ and $\Gamma = \Gamma_0$. Our goal is to obtain asymptotic estimates for the accuracy of the PML solution in two limits. The first is for a fixed PML as the lateral distance $|x_1| \to \infty$ and the second is for a fixed position as the imaginary part of $\tilde{L}$ increases. A corollary of the first estimate is a lower bound for the error in the PML solution in the V_H norm showing that

If we now replace (2.10) by the Dirichlet condition that $u_p = 0$ at $x_2 = L + H$ we obtain the following solution valid for $0 \le x_2 < y_2$ and denoted $u_{p,d}$:

$$
u_{p,d}(x) = \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \dfrac{\sin(\kappa x_2)}{\kappa} \dfrac{\sin(\kappa((\tilde{L}+H)-y_2))}{\sin(\kappa(\tilde{L}+H))} \exp(-\mathrm{i}\xi x_1)\, d\xi, & 0 \le x_2 < y_2, \\ \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \dfrac{\sin(\kappa y_2)}{\kappa} \dfrac{\sin(\kappa((\tilde{L}+H)-x_2))}{\sin(\kappa(\tilde{L}+H))} \exp(-\mathrm{i}\xi x_1)\, d\xi, & y_2 < x_2 \le H. \end{cases} \tag{4.5}
$$

We will find second representations for the exact solutions to the PML problems useful. If $|x_1|$ is large enough, precisely if $(H + \tilde{L})|x_1| > (x_2 + y_2) \tilde{L}$, we can evaluate the integrals (4.4) and (4.5) exactly, as a residue series, by contour integration, e.g. by moving the path of integration to the path $\xi = Y$, for some $Y < 0$, and then letting $Y \to \infty$

while, for $k|H+\tilde{L}|\geq\pi$ *and*

$$\frac{3\sqrt{5}\pi(H+\tilde{L})|x_1|}{40k|H+\tilde{L}|^2}\geq\frac{|H+\tilde{L}|^2}{\pi\tilde{L}}+x_2$$

and

(4.15) $R^2 = (1 - n^2\pi^2(q^2 + p^2))^2 + 4n^2\pi^2 q^2 = (1 - |\alpha_n|^2)^2 + 4n^2\pi^2 q^2 \leq (1 - |\alpha_n|^2)^2 + 4|\alpha_n|^2 = (1 + |\alpha_n|^2)^2.$

Thus, for p, q, and n all small,

$$\beta_n \approx pqn^2\pi^2,$$

while, for all values of n, p and q,

$$\beta_n \geq \frac{pqn^2\pi^2}{\sqrt{R}} \geq \delta_n := \frac{pqn^2\pi^2}{\sqrt{1 + |\alpha_n|^2}} = \frac{pqn^2\pi^2}{\sqrt{1 + n^2\pi^2(p^2 + q^2)}}.$$

Now

$$\begin{aligned}
\frac{\delta_{n+1} - \delta_n}{pq\pi^2} &= \frac{(n+1)^2\sqrt{1 + n^2\pi^2(p^2+q^2)} - n^2\sqrt{1 + (n+1)^2\pi^2(p^2+q^2)}}{\sqrt{1 + (n+1)^2\pi^2(p^2+q^2)}\sqrt{1 + n^2\pi^2(p^2+q^2)}} \\
&\geq \frac{(n+1)^4(1 + n^2\pi^2(p^2+q^2)) - n^4(1 + (n+1)^2\pi^2(p^2+q^2))}{2(n+1)^2(1 + (n+1)^2\pi^2(p^2+q^2))\sqrt{1 + n^2\pi^2(p^2+q^2)}} \\
&= \frac{(2n+1)n^2(2 + (n+1)^2\pi^2(p^2+q^2))}{2(n+1)^2(1 + (n+1)^2\pi^2(p^2+q^2))\sqrt{1 + n^2\pi^2(p^2+q^2)}} \\
&\geq \frac{3n}{8\sqrt{1 + n^2\pi^2(p^2+q^2)}} \geq \frac{3}{8\sqrt{1 + \pi^2(p^2+q^2)}},
\end{aligned}$$

It follows that

$$\delta_n \geq \delta_1 + (n-1)\frac{3pq\pi^2}{8\sqrt{1 + \pi^2(p^2+q^2)}}$$

and so

$$Q_n \leq k(x_2 + y_2)\pi q - \frac{k|x_1|pq\pi^2}{\sqrt{1 + \pi^2(p^2+q^2)}} + (n-1)\Psi = -\frac{5k|x_1|pq\pi^2}{8\sqrt{1 + \pi^2(p^2+q^2)}} + n\Psi,$$

where

$$\Psi := k(x_2 + y_2)\pi q - \frac{3k|x_1|pq\pi^2}{8\sqrt{1 + \pi^2(p^2+q^2)}}.$$

Noting also that, from (4.15),

$$|\beta_n| = \sqrt{R} \geq \sqrt{2\pi q n} \geq$$

where $S_{H,A}$

We note that $B \to \infty$ as $k|\tilde{L}| \to \infty$. Moreover, defining

$$A := 2\theta^{-1} \log B, \tag{4.25}$$

we see that, for $k|\tilde{L}| \quad 1$, both (4.22) and (4.24) are satisfied and, moreover $kA \quad 1$ and $kA \quad (kH)^2$. Thus we have shown that (4.21) holds with A given by (4.25) if $k|\tilde{L}|$ is sufficiently large, i.e. we have shown the following bound on $u - u_{p,d}$. The bound in this theorem on $u - u_{p,n}$ follows completely analogously, with minor changes to numerical values.

Theorem 4.2. *Provided $k|\tilde{L}|$ is sufficiently large, it holds that*

$$u - u_{p,d} \quad {}_{V_H} \geq \hat{c}k^{1/2}y_2H^{3/2}(H + \quad \tilde{L}) \quad \tilde{L}$$

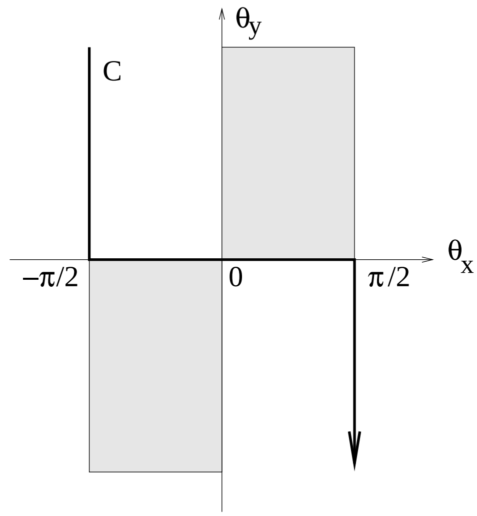

Fig. 4.1. *A schematic showing the contour C in the complex θ plane ($\theta = \theta_x + \mathrm{i}\theta_y$) together with the unbounded regions (shaded) in which the poles of the integrand in $-\pi/2 \le \theta_x \le \pi/2$ may lie.*

Now, applying (3.2), as at the end of the proof of Theorem 3.1, we see that |1 + exp(2i $\sqrt{}$

Our next result is more optimistic than the previous ones. For any fixed point x

Note that the integrand is analytic except for poles that lie in the shaded regions in Fig. 4.1 (but not on their boundaries). Now, writing $\theta = \theta_x + \mathrm{i}\theta_y$, we note that as $\theta_y \to +\infty$ with $-\pi/2 \le \theta_x \le 0$, $\exp(2\mathrm{i}k(\tilde{L} + H)\cos\theta) \to 0$, uniformly in θ_x, since $(\tilde{L}) > 0$ and $(\tilde{L}) > 0$. Thus, for $-\pi/2 \le \theta_x \le 0$ and all θ_y sufficiently large, the modulus of the integrand in (4.31) is $\le 2\exp(D(\theta))$, where

$$D(\theta) = k|\sinh\theta_y| \left((x_2 + y_2 - 2((\tilde{L}) + H))|\sin\theta_x| + (|x_1| - 2 (\tilde{L}))\cos\theta_x \right).$$

The same bound holds for $0 \le \theta_x \le \pi/2$ if θ_y is sufficiently large and negative. Thus, for $|x_1| < 2 (\tilde{L})$, we can deform the contour C to the imaginary axis, i.e. to the contour $\theta = -\mathrm{i}t$, $-\infty < t < +\infty$, so that

$$\begin{aligned} &u(x) - u_{p,d}(x) \\ &= \frac{-1}{\pi} \int_{-\infty}^{\infty} \exp(kx_1 \sinh t) \sin(kx_2 \cosh t) \sin(ky_2 \cosh t) \frac{\exp(2\mathrm{i}k(\tilde{L} + H)\cosh t)}{1 - \exp(2\mathrm{i}k(\tilde{L} + H)\cosh t)}\, dt. \end{aligned}$$

Now using the substitution $\cosh t = 1 + u^2$ and $\sinh t = u\sqrt{2 + u^2}$ we obtain

$$\begin{aligned} &u(x) - u_{p,d}(x) \\ &= \frac{-2}{\pi} \int_{-\infty}^{\infty} \exp(kx_1 u\sqrt{2 + u^2}) \sin(kx_2(1 + u^2)) \sin(ky_2(1 + u^2)) \\ &\qquad \frac{\exp(2\mathrm{i}k(\tilde{L} + H)(1 + u^2))}{1 - \exp(2\mathrm{i}k(\tilde{L} + H)(1 + u^2))} \frac{du}{\sqrt{2 + u^2}} \end{aligned}$$

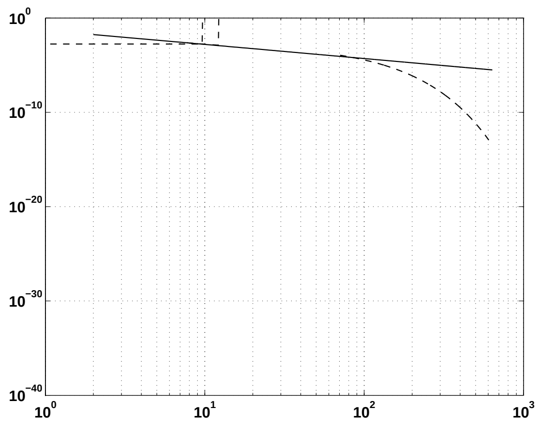

Fig. 4.2.

$f \quad H^{-1/2}(\Gamma_{H+L})$ is a given function. We want to study the problem of finding $v \quad V_{H+L}$ such that

$$\frac{1}{\sigma}\frac{\partial v}{\partial x_n} = f \text{ on } \Gamma_{H+L}, \tag{5.2}$$

$$\Delta_p v + k^2 \sigma v = g \text{ in } S_{H+L}. \tag{5.3}$$

Proceeding as for the simple PML, we see that $v \quad V$

As argued earlier in this section, if $\tilde{L}$ is chosen appropriately we can ensure that the above problem has a unique solution for each n and so the iteration is well defined. Of course in practice we shall use a finite element approximation of the boundary value problem of finding $u^{(n)} \in V_{H+L}$ such that

$$\frac{1}{\sigma}\frac{\partial u^{(n)}}{\partial x_n} = E_p u^{(n-1)} \text{ on } \Gamma_{H+L}, \tag{5.5}$$

$$\Delta_p u^{(n)} + k^2 \sigma u^{(n)} = g \text{ in } S_{H+L}. \tag{5.6}$$

Thus at each iteration we must evaluate E_p (using the Fast Fourier Transform) and then solve a finite element problem on the strip S_{H+L} (in practice, truncated laterally). More details of one possible finite element method are given in the next section.

We now want to investigate the convergence of the scheme. Using the PML inf-sup condition and the trace estimate (3.1),

$$\begin{aligned}
\gamma_p \|u - u^{(n)}\|_V &\le \sup_{\varphi \in V_H} \frac{|b(u - u^{(n)}, \varphi)|}{\|\varphi\|_{V_H}} \\
&= \sup_{\varphi \in V_H} \frac{\left|\int_{\Gamma_H} \bar{\varphi} N_p E_p (u - u^{(n-1)})\, ds\right|}{\|\varphi\|_{V_H}} \\
&\le 2\|N_p E_p\|_{L(H^{1/2}(\Gamma_H), H^{-1/2}(\Gamma_H))} \|u - u^{(n-1)}\|_{V_H}.
\end{aligned}$$

It remains to estimate $\|N_p E_p\|_{L(H^{1/2}(\Gamma_H), H^{-1/2}(\Gamma_H))}$. From the Fourier representation of N_p and E_p we see that, in the Fourier domain, the action of $N_p E_p$ corresponds to multiplication by

$$z_{NE} = \frac{2z\exp(-z\tilde{L})}{\exp(z\tilde{L}) + \exp(-z\tilde{L})}.$$

To estimate the operator norm it therefore suffices to bound

$$\max_{\xi \in \mathbb{R}} \frac{|z_{NE}(\xi)|}{\sqrt{k^2 + \xi^2}}.$$

But this has already been done in Theorem 3.1 and we conclude that

$$\|N_p E_p\|_{L(H^{1/2}(\Gamma_H), H^{-1/2}(\Gamma_H))} \le C_U(\alpha, \beta).$$

Using the estimate (5.1) for γ_p this implies that

$$\|u - u^{(n)}\|_{V_H} \le \frac{2C_U(\alpha,\beta)}{\gamma - 2C_U(\alpha,\beta)} \|u - u^{(n-1)}\|_{V_H}. \tag{5.7}$$

The constant $2C_U(\alpha,\beta)/(\gamma - 2C_U(\alpha,\beta))$ can be made less than one by choosing α and β large enough, while retaining some constraint on the ratio α/β, and in that case the iterative scheme will converge. Note that this is less restrictive than having to choose α and β so that

$$2C_U(\alpha,\beta)/(\gamma - 2C_U(\alpha,\beta)) < \epsilon$$

which is required to ensure a relative error ϵ by our convergence result in Theorem 3.3. Thus the PML can be thinner. The price to be paid for the thinner PML is that at each iteration we must compute the action of the operator E_p. But this is not a singular integral operator and the action can be computed efficiently via the Fast Fourier Transform as we shall see in the next section.

In summary we have proved the following theorem.

Theorem 5.1. *Suppose D satisfies the boundary constraint (1.8). If σ is chosen so that $4C_U(\alpha,\beta) < \gamma$ then the iterative scheme defined by (5.4) is well defined and $u^{(n)}$ converges linearly to the exact solution u according to (5.7).*

6. Numerical results. So far we have assumed that the data g is supported in S_H. This can be inconvenient since we want to take H as small as possible in order to decrease the thickness of the region to be covered by finite elements (or we may wish to use a point source that is not in V_H). To handle this case we define the incident field denoted u_h^I by

$$u_h^I(x) = -\int_D G_h(x,y)g(y)\,dy \text{ for } x \in U_h,$$

where G_h is the Dirichlet Green's function for the half-space U_h above Γ_h for some $h < f_-$. In this case

$$G_h(x,y) = \Phi(x,y) - \Phi(x,y_h)$$

where, if $y = (\tilde{y}, y_n)$, the reflected point $y_h = (\tilde{y}, 2h - y_n)$, and $\Phi(x,y) := \frac{\mathrm{i}}{4}H_0^{(1)}(k|x-y|)$ in 2D, $:= \exp(\mathrm{i}k|x-y|)/(4\pi|x-y|)$ in 3D, is the standard fundamental solution of the Helmholtz equation. Restricting attention to the case when Γ is Lipschitz, specifically the graph of a bounded and uniformly Lipschitz function, in which case there exists a well-defined trace operator from $H^1(D)$ to $H^{1/2}(\Gamma)$, we can allow in

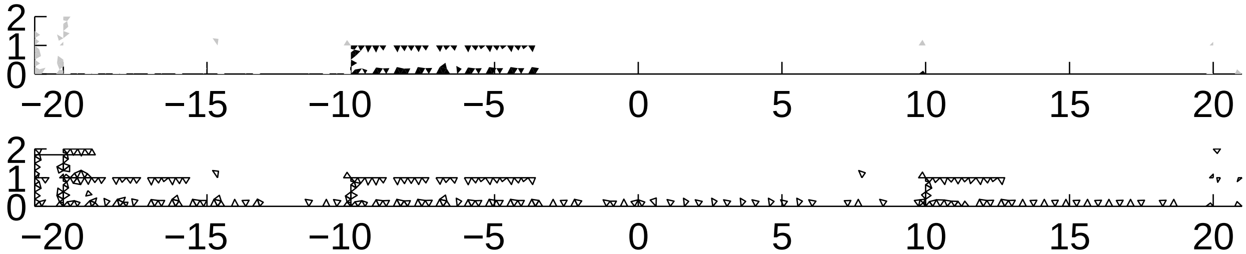

Fig. 6.1. *The top panel shows the domain of the computation. The gray region is PML. The black and white regions are the truncated S_H and the black region is where the error between the exact and PML/FEM solution is computed. The lower figure shows the mesh which is very fine since we wish to study effects of the PML rather than the mesh.*

6.1. A flat scatterer. Here we provide some numerical tests of the straightfoward use of the PML to terminate the model problem of computing the solution of scattering of the field due to a point source above an infinite flat boundary. Of course this is a special case, but it has the advantage that we know the exact solution.

In this case $D = U_0$ and $\Gamma = \{(x_1, 0) : x_1 \in \mathbb{R}\}$. The point source is located at $y = (0, y_2)$, $y_2 > 0$. Using the image principle it is then immediate that the total field in U_0 is

$$u(x) = \Phi(x, y) - \Phi(x, y^*)$$

where $y^* = (0, -y_2)$ is the image point.

We choose as an incident field

$$u^i(x) = \Phi(x, y) - \Phi(x, y_h) \tag{6.8}$$

where $y_h = (0, 2h - y_2)$ and $h < 0$ is a parameter and u^i is analytic in a neighborhood of Γ. This incident field has the same decay as the solution as $|x_1| \to \infty$. The exact scattered field is

$$u^s(x) = u(x) - u^i(x) = \Phi(x, y_h) - \Phi(x, y^*).$$

The computational domain is truncated laterally at $x_1 = -A$ and $x_1 = A$ using a PML of width L in the $\pm x_1$ directions. This aspect of the truncation procedure is not captured by our preceding analysis. For a simple model problem we choose the parameter values given in the following table. The PML parameter σ is given by (2.8) in $|x_1| \leq A$ and by the same formula with

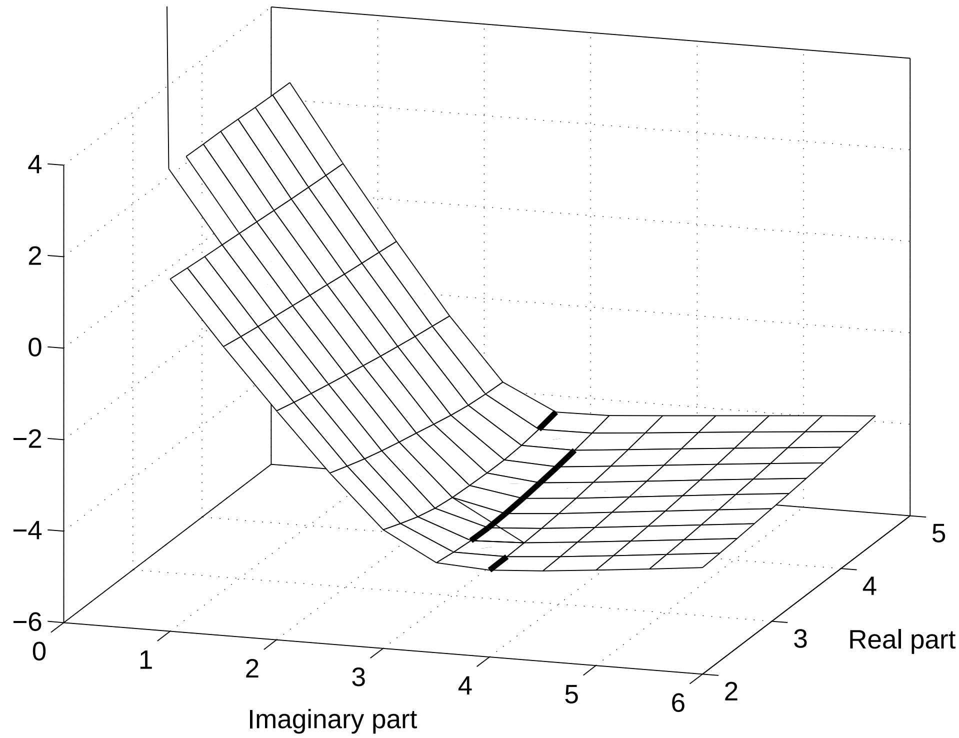

Fig. 6.2. *A surface plot of the base 10 logarithm of the discrete relative L^2 error for the PML/finite element solution against the real and imaginary parts of $\tilde{L}$ (using a Neumann boundary condition on the PML). The solid line marks the minimum error for fixed real part as the imaginary part is varied. Clearly this graph suggests exponential convergence as the imaginary part of σ increases and almost no dependence on the real part.*

6.2. The iterative scheme. Next we shall test the convergence of the iterative scheme described in

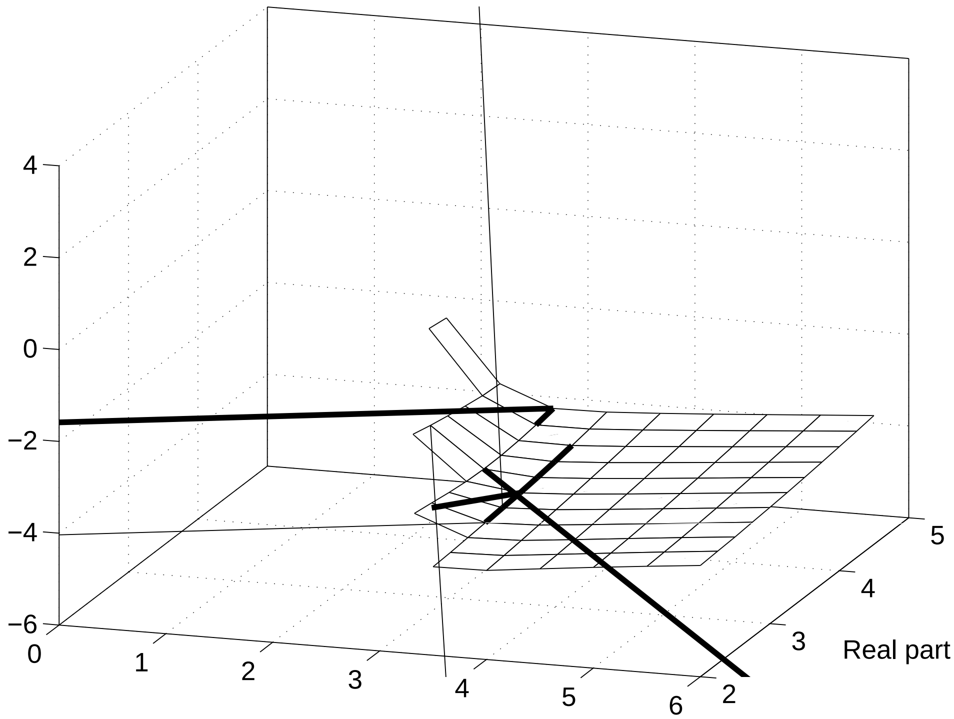

Fig. 6.3. *A surface plot of the base 10 logarithm of the discrete relative L^2 error for the PML/finite element solution against the real and imaginary parts of $\tilde{L}$ using the homogeneous Dirichlet boundary condition on the PML. Generally the results are similar to those computed using the Neumann boundary condition in Fig. 6.2.*

Iteration number	Relative discrete error
1	24.8%
2	5.54%
3	3.69%
4	4.33%

Table 6.1
Error in successive iterates of the iterative correction scheme. After $n = 4$ the error stagnates at about 4%

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pick a priori, we have also proposed a simple iterative scheme to correct the PML solution, proving linear convergence, which we illustrate with numerical results for a simple case.

Three important questions are unanswered by this study:

1. How is the method influenced by lateral termination?
2. What is the error for the finite element method applied to the truncated problem (the di□culty is to obtain the dependence of the error on the PML parameters and lateral cuto□)?
3. In the general case, is convergence exponential on compact subsets of S_H?.

Finally we have not addressed the practical problem of how to solve the linear system resulting from the

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[6]