Periodic solutions for nonlinear dilation equations

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Abstract

We consider a class of functional equations representing nonlinear dilation maps of the real line having an invariant interval bounded above by a fixed point. Necessary and sufficient conditions for the existence of periodic solutions demand that the maps satisfy an eigenproblem, with integer eigenvalues, for a certain nonlinear generalisation of Chebyschev's ordinary differential equation. Hence we obtain generalisations of Chebyschev polynomials, where the associated functional equation has

$\lambda = \lambda_1 = \lambda_0^{1/s}$. So setting $s = 2/(q+1)$ we see that $\varphi_1'(x) \quad x$ for small x, and hence $\varphi_1''(0) = 0$, and is therefore strictly negative (since one is an upper bound).

Conversely if $\varphi_1(x)$ is a solution of (1), for which $\lambda = \lambda_1$

Proof It is straightforward to show by induction that $\varphi_n(0) = 1$, $\varphi_n'(0) = 0$, $\varphi_n''(0) = -1$, and φ_n is even for all n. Therefore we show that the sequence converges: the rest follows immediately. Applying the mean value theorem

$$|\varphi_{k+1}(x) - \varphi_k(x)| = |F^{(k)}(F(\varphi_0(\frac{x}{\lambda^{k+1}}))) - F^k(\varphi_0(\frac{x}{\lambda^k}))|$$

$$= |\frac{dF^{(k)}}{dx}(\theta)||F(\varphi_0(\frac{x}{\lambda^{k+1}})) - \varphi_0(\frac{x}{\lambda^k})|,$$

for some θ between $\varphi_0(x/\lambda^k) = 1 - \frac{x^2}{2\lambda^{2k}}$ and $F(1 - \frac{x^2}{2\lambda^{2(k+1)}})$. The first factor behaves like $F'(1)^k = \lambda^{2k}$ as $k \to \infty$; and second factor behaves like $F''(1)x^4/4\lambda^{4(k+1)}$ as $k \to \infty$. Hence

$$|\varphi_{k+1}(x) - \varphi_k(x)| \to 0$$

uniformly on [-1,1] and the result follows.

The curve $(y, F(y))$ remains within the box $[-1, 1] \times [-1, 1]$: yet $\varphi(x)$ may be periodic or wandering. For example if $F(n) = T_n(y)$ then $\varphi(x) = \cos(x)$ is 2π periodic. However next we show that cases such as these are nongeneric.

Suppose the solution φ is P-periodic, satisfying $\varphi(x + P) = \varphi(x)$ for all x $[0, P]$, with some minimal period P (φ is not periodic for any smaller period, P'). Then we have, for all x,

$$\varphi(\lambda P + x) = F(\varphi(P + x/\lambda$$

(1) and (2). For such a periodic solution, $\varphi(x)$, this requires that

$$F(y) = \varphi(n\varphi^{-1}(y))$$

is well defined considering all branches of φ^{-1}. Next we give a su□cient condition on F.

Theorem 3 *Let $\varphi(x)$ be a twice continuously di erentiable periodic function with range $[\beta, 1]$ (for some constant $\beta < 1$), satisfying*

$$\varphi(0) = 1, \text{ and } \varphi'(0) = 0$$

together with the equation

$$\varphi''(x) = \dot{G}(\varphi(x))/2, \tag{4}$$

for some smooth nonnegative function $G : [\beta, 1] \to \mathrm{R}^+$, where $\dot{G}(w)$ denotes the derivative of $G(w)$ at w, and satisfying $\dot{G}(1) = -2$, $G(\beta) = G(1) = 0$.

Then for any integer n, if F and φ also satisfy (1), for $\lambda = n$, (and (2)) then F is the solution of the di erential equation

n^2 (4)(w)

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through by $\frac{dF}{dy}(y)$, we obtain

$$n^2 G(F(y)) = G(y)\left(\frac{dF}{dy}\right)^2.$$

If we write $F(y) = f(x)$ where $y = \varphi(x)$, this last becomes

$$n^2 G(f) = \left(\frac{df}{dx}\right)^2.$$

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