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Unique Extension of Atomic Functionals of JB*-Triples

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Declaration

I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

Martin Hunt

Abstract

This thesis initiates and proceeds to develop a theory of unique norm

Acknowledgements

Contents

property for C*-algebras given in [13], the extreme extension property for JB*-triples does not imply the Cartan extension property.

The thesis is organised as follows.

Chapter 1 contains no new results. Its purpose is to collect together the well-known background of JB*-triples which is fundamental for the understanding of the remainder of the thesis. Chapter 1 gives priority to key results and concepts which will be used frequently in the sequel. Known material which is used infrequently is introduced in the place where it is first needed.

Chapter 2, which is largely expository, contains Banach space theory appropriate to the theme of this thesis and sets the scene. In addition to some necessary background material, the second chapter also contains a brief discussion of Banach spaces X for which all closed subspaces have the extension property in X. It is pointed out that Hilbert spaces are the only JB*-triples with this property. The chapter concludes with a brief discussion of c_0-sums of closed subspaces of a Banach space X in relation to the extreme extension property.

The focus of Chapter 3 is on unique norm preserving extension of predual elements of Cartan factors. When C is a Cartan factor and JBW*-subtriple of a Cartan factor D it is shown (Theorem 3.3.4) that the existence of a single element ρ of $\partial_e(C_{*,1})$ with unique extension in $\partial_e(D_{*,1})$ implies that the same is true of *every* element of $\partial_e(C_{*,1})$ and analysis is made of Cartan factor inclusions that force this property, summarised in Theorem 3.3.11.

When this partial unique extension condition compels C to be an inner ideal is considered and conclusions drawn (for example, Theorem 3.4.8). Turning attention to norm one non-extreme elements in the predual of C, one of our key results (Theorem 3.5.6) is that unique norm one extension of just one of these forces C to be an inner ideal. The chapter concludes with applications to von Neumann algebras.

Extreme points of the dual ball of a JB*-triple live on Cartan factors contained in the atomic part of the second dual, allowing the work of Chapter 3 to be brought to bear, in Chapter 4, upon unique extension theory of elements of $\partial_e(A_1^*)$ to $\partial_e(B_1^*)$ when A is a JB*-subtriple of a JB*-triple B and consequently upon the situation when A has the extreme extension property in B, and connections with representation theory is made. The *Cartan extension property* is introduced and characterisations given in terms of a stronger unique extension property and of inner ideal structure of atomic parts (Theorem 4.4.5). In an extension of Theorem 3.5.6, a main conclusion of Chapter 4 is that a norm one atomic functional has a unique extension to a norm one atomic functional if and only if it has a unique norm one extension (Theorem 4.5.1) and a discussion of the atomic extension property evolves. A brief review of repercussions for appropriate *state* extensions in the category of JB*-algebras is provided.

Chapter 5 culminates in a solution of the extreme, Cartan and atomic extension properties of a *separable* JB*-subtriple in a JBW*-triple and, in so doing, extends C*-algebra work of [13], [18] to the JB*-triple setting. To acquire this solution we are lead into different areas, obtaining a number of independently interesting results in Banach space properties of JB*-triples along the way. In particular, for every JB*-triple A it is shown that weak sequential convergence in $\partial_e(A_1^*)$ coincides with norm convergence (Theorem 5.3.7). When A is *separable*, exploiting an important result in [19], we deduce that weak* sequential convergence and norm convergence coincide in $\partial_e(A_1^*)$ precisely when A is a weakly compact JB*-triple. This enables the above mentioned solution. The chapter begins with a discussion of weakly compact JB*-triples in which a new 'inner ideal' equivalence is included.

Chapter 1

Jordan Structures

1.1 Introduction

The papers [38], [39] and [41] of Friedman and Russo serve as a good introduction to the properties of JB*-triples. 'Concrete' JB*-triples, now known as JC*-triples, were first studied by Harris in [45] and [46] under the name J*-algebras. Although now firmly a branch of functional analysis and operator algebra, JB*-triples have their origin in the theory of bounded symmetric domains. The equivalence of JB*-triples and Banach spaces whose open unit ball is a bounded symmetric domain is due to Kaup [52]. A survey and history of this area (not discussed in this thesis) is given in [20].

The reader is referred to the books of Pedersen [61] and Rudin [63] for the basics of functional analysis. Relevant material on C*-algebras can be found in [58] and [60].

General mathematical notation used throughout this thesis is standard. If X is a Banach space, X_1 and $S(X_1)$ denote the closed unit ball and unit sphere of norm one elements, respectively. In the usual way, X is regarded as being a subspace of X^{**} and X^* is identified with the weak* continuous linear functionals on X^{**}. Our usage of 'σ-convex sum' includes 'convex sum'. Thus a σ-convex sum of a Banach space X is a finite or infinite series $\sum \lambda_n x_n$ where $\lambda > 0$ for all n and $\sum \lambda_n = 1$. If K is a convex set $\partial_e(K)$ denotes the set of extreme points of K. For a locally compact Hausdorff space S, $C_0(S)$ stands for the continuous functionals vanishing at infinity. The notations R, C and H stand for the real numbers, complex numbers and the quaternions, respectively.

1.2 Algebras

By an *algebra* shall be meant a complex linear space A together with a bilinear map, $\pi : A \times A \to A$, referred to as a *product*. If accompanied by a conjugate linear map of order two, $a \to a^*$, such that

$$(\pi(a, b))^* = \pi(b^*, a^*)$$

then, with respect to π, $a \to a^*$ is an *involution* and A is a **-algebra*. The powers of an element a may be defined inductively by

$$a^{n+1} = \pi(a^n, a), \qquad \text{for all } n \geq 1.$$

In this generality no assumption is made concerning commutativity nor associativity of products. In particular, the above definition of powers is one-sided and we need not have $(a^n)^* = (a^*)^n$, in the case of a *-algebra.

1.3 Jordan Algebras

An algebra A with product, $(a, b) \to a \circ b$, is said to be a *Jordan algebra* if

$$a \circ b = b \circ a \quad \text{and} \quad a \circ (b \circ a^2) = (a \circ b) \circ a^2.$$

The associated *Jordan triple product* is

$$\{a \circ b \circ c\} = (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b.$$

Jordan algebras are commutative but not necessarily associative. An associative Jordan algebra is said to be *abelian*.

Let a belong to A, where A is a Jordan algebra. We have, for $m, n \geq 1$,

$$a^{m+n} = a^m \circ a^n,$$

from which it follows that the Jordan subalgebra generated by a is abelian.

The multiplication operator T_a and quadratic operator U_a are defined on A by

$$T_a(b) = a \circ b \quad \text{and} \quad U_a(b) = \{a \circ b \circ a\}.$$

An *ideal* of a Jordan algebra A is a linear subspace I for which $A \circ I$ is contained in I. In which case, A/I is canonically a Jordan algebra and I is, in particular, a subalgebra of A. A *quadratic ideal* of a Jordan algebra A is a linear subspace I for which $U_a(I)$ is contained in A, for all elements a in A.

1.4 Complex Jordan *-Triple Systems

1.4.1 Throughout this thesis a *Jordan *-triple system* shall mean a complex linear space A together with a triple product, $\{\cdots\} : A^3 \to A$, linear and symmetric in the outer two variables, conjugate linear in the middle variable and satisfying the *main identity*

$$\{ab\{xyz\}\} = \{\{abx\}yz\} + \{xy\{abz\}\} - \{x\{bay\}z\}.$$

Given a and b in A, where A is a Jordan *-triple system, the operators $D(a, b)$ and $Q_{a,b}$ on A

Other useful identities (an extensive list can be found in [57]) are

(i) $D(x, y)Q_x = Q_x D(y, x)$

(ii) $D(Q_x(y), y) = D(x, Q_y(x))$

(iii) $Q_{Q_x(y)} = Q_x Q_y Q_x$

(iv) $[D(x, y), D(a, b)] = D(\{xya\}, b) - D(a, \{bxy\})$, where $[\cdot, \cdot]$ denotes the commutator.

By means of the above identity (b), and separately that of (iv) – the latter in conjunction with liberal use of the fact that the triple product is symmetric in the outer variables, we have the following.

Proposition 1.4.3

*The following are equivalent for a Jordan *-triple system A.*

(a) $D(x, x)D(y, y) = D(y, y)D(x, x)$, *for all* $x, y \in A$.

(b) $D(x, y)D(a, b) = D(a, b)D(x, y)$, *for all* $a, b, x, y \in A$.

(c) $\{xy\{abc\}\} = \{x\{yab\}c\}$, *for all* $a, b, c, x, y \in A$.

A Jordan *-triple system satisfying any of the equivalent conditions of Proposition 1.4.3 is said to be *abelian*.

Given an element a in a Jordan *-triple system A, the odd 'powers' of a are defined inductively by

$$a^1 = a, \quad a^{(2n+1)} = \{aa^{(2n-1)}a\}, \quad n \geq 1.$$

The set of all odd powers of a is an abelian subtriple of A, the Jordan *-triple in A generated by a.

Let $\pi : A \to B$ be a linear map between Jordan *-triple systems. If

$$\pi(\{abc\}) = \{\pi(a)\pi(b)\pi(c)\}, \quad \text{for all } a, b, c \in A$$

then π is said to be a *triple homomorphism*. In fact, by polarisation, π is a triple homomorphism if

$$\pi(\{aaa\}) = \{\pi(a)\pi(a)\pi(a)\}, \quad \text{for all } a \in A.$$

1.4.4 Significant classes of Jordan *-triple systems arise from Jordan *-algebras and associative *-algebras in a way now described.

(a) Let A be a complex Jordan *-algebra. Then A is a Jordan *-triple system with respect to the triple product

$$\{abc\} \;=\; \{a \circ b^* \circ c\} \;=\; (a \circ b^*) \circ c + a \circ (b^* \circ c) - (a \circ c) \circ b^*.$$

(b) If A is an associative complex *-algebra, so that A is a special Jordan *-algebra via the special Jordan product $a \circ b = \frac{1}{2}(ab + ba)$, then A is a Jordan *-triple system via

$$\{abc\} \;=\; \frac{1}{2}(ab^*c + cb^*a).$$

We note that (b) is a special case of (a).

1.4.5 Let I be a linear subspace of a Jordan *-triple system A. In order of increasing generality, I is said to be

(a) an *ideal* of A if $\{IAA\} + \{AIA\}$ is contained in I;

(b) an *inner ideal* of A if $\{IAI\}$ is contained in I;

(c) a *subtriple* of A if $\{III\}$ is contained in I.

By the polarisation identities of 1.4.2 the conditions (b) and (c) are respectively equivalent to

(b′) $\{xAx\}$ I, for all x I;

(c′)

1.4.6 Let A

We write $e \perp f$ when e and f are orthogonal and $e \leq f$ when $f - e$ is a tripotent orthogonal to e.

Since, for $j \in \{0, 1, 2\}$, $f \in A_j(e)$ if and only if $2\{eef\} = jf$, the

1.5 JB*-Algebras

1.5.1 A *JB*-algebra* is a Jordan *-algebra and Banach space A satisfying

The latter set is the spectrum of $a \in A_{sa}$ and coincides with the set

$$\{\lambda \in \mathbb{C} : a - \lambda 1 \text{ is not invertible in } \tilde{A}\}.$$

The class of JB*-algebras includes all C*-algebras. A C*-algebra A is a JB*-algebra via the special Jordan product

$$a \circ b = \frac{1}{2}(ab + ba)$$

and A_{sa} is a JB-algebra. A *JC*-algebra* is a JB*-algebra that is (isometrically) Jordan *-isomorphic to a norm closed Jordan *-subalgebra of a C*-algebra. The self-adjoint part of a JC*-algebra is called a *JC-algebra*. The canonical example of a JB*-algebra that is not a JC*-algebra is the exceptional algebra M_3^8 of the 3×3 hermitian matrices over the complex octonions (see 1.11.1 (6)).

A *JW*-algebra* is a JC*-algebra with a predual. The second dual A^{**} of a JC*-algebra A is a JW*-algebra. The following Gelfand-Naimark theorem shows that every JBW*-algebra decomposes into an orthogonal sum of 'special' and 'exceptional' weak* closed ideals.

Theorem 1.5.2 [66, 3.9]
Every JBW-algebra is an ℓ_∞-sum of the form*

$$M \oplus C(X, M_3^8)$$

where M is a JW-algebra and X is a compact hyperstonean space. Hence, every JB*-algebra is weak* dense in a JBW*-algebra of this form.*

1.5.3 The set of positive elements of a JB*-algebra A,

$$A_+ = \{a^2 : a \in A_{sa}\},$$

is stable under addition and by multiplication by positive scalars. Moreover, $A_{sa} = A_+ - A_+$. The set of positive linear functionals on A is

$$A^*_+ = \{\phi \in A^* : \phi(A_+) \subseteq [0, \infty)\}.$$

The *quasi-state space* of A,

$$Q(A) = \{\phi \in A^*_+ : \|\phi\| \leq 1\} = A^*_+ \cap A^*_1,$$

is a convex weak*-compact subset of A^*_1. The *state space* of A

$$S(A) = \{\rho \in A^* : \rho(1) = 1 = \|\rho\|\},$$

where 1 is the identity element of A^{**}, is the convex set of positive linear functionals of norm 1. We have

$$\partial_e(Q(A)) \setminus \{0\} = \partial_e(S(A)).$$

This latter set is the set of *pure states* of A and is denoted by $P(A)$.

A positive linear functional ϕ on a JB*-algebra A is said to be *faithful* if

$$(\ker \phi) \cap A_+ = \{0\}$$

The *normal states* of a JBW*-algebra M are the weak*-continuous states of M.

1.6 JB*-Triples

A *JB*-triple* is a complex Banach space and a Jordan *-triple system A such that

(a) $\|\{aaa\}\| = \|a\|^3$, for all $a \in A$;

(b) for each $a \in A$, the operator $D(a, a)$ is hermitian with non-negative spectrum.

Every norm closed subtriple of a JB*-triple is a JB*-triple. A *JBW*-triple* is a JB*-triple M with a predual M_*. By [48, 3.21, 3.24] and [6, 2.1], such a predual is unique and the triple product on a JBW*-triple is separately weak* continuous in each variable. The second dual of a JB*-triple is a JBW*-triple containing A as a JB*-subtriple [26].

If e is a tripotent in a JBW*-triple M the Peirce subspaces $M_j(e)$ are JBW*-subtriples of M for $j = 0, 1$ and 2, and $M_2(e)$ is a JBW*-algebra with the product and involution given in 1.4.4.

A fundamental result concerning triple homomorphisms is the following.

For any pair of complex Hilbert spaces H and K the Banach space of operators $B(H, K)$ with triple product

$$\{abc\} = \frac{1}{2}(ab^*c + cb^*a),$$

is realised(1)]TJt24a6355Tf-251.74-33.624Td[(is)-3JW*-sub(duct)]TJ6x cb

1.7 Local JB*-algebra structure

If x is an element of a JB*-triple A, A_x and $A(x$

The classification of norm closed inner ideals in a JB*-triple by unique extensions is fundamental.

Theorem 1.8.3 [33, 2.5, 2.6]
Let I be a JB-subtriple of a JB*-triple A. Let J be a JBW*-subtriple of a JBW*-triple M.*

(a) *I is an inner ideal of A if and only if each ρ in I^* has unique norm preserving extension in A^*.*

(b) *J is an inner ideal of M if and only if each ρ in J_* has unique norm preserving extension in M_*.*

The following is an amalgamation of [30, §5] and [35, §4].

Theorem 1.8.6
Let M be a JBW-triple.*

(a) *Every structural projection on M is weak* continuous and contractive.*

(b) *There is a bijection from the set of all structural projections of M onto the set of all weak* closed inner ideals of M given by*

$$P \longrightarrow P(M).$$

1.9 Orthogonality

1.9.1 The notion of orthogonality of tripotents (see remarks following Proposition 1.4.9) in JB*-triples generalises.

Let A be a JB*-triple. Elements a and b in A are said to be *orthogonal* if

$$D(a, b) = 0.$$

($D(a, b) = 0$ if and only if $D(b, a) = 0$ [34, 3.1].) Let a, b A and S and T be subsets of

Theorem 1.9.2

Let A be a JB-triple. Let S be a subset of A, B and C be JB*-subtriples of A, I a norm closed inner ideal of A and let J be a norm closed ideal of A.*

(a) *$S^{\perp}$ is a norm closed inner ideal of A, and is an ideal of A if S is an ideal of A. Moreover, $S \subseteq S^{\perp\perp}$.*

(b) *$A_2(e)^{\perp} = A_0(e)$ for each tripotent e in A.*

(c) *If B and C are orthogonal (that is, $B \perp C$), then*

$$B + C = B \oplus_{\infty} C$$

is a JB-subtriple of A containing B and C as norm closed ideals.*

(d) *$I \cap J = \{0\}$ if and only if $I \perp J$.*

(e) *If A is a JBW*-triple and J is weak* closed then*

$$A = J + J^{\perp} \quad (= J \oplus_{\infty} J^{\perp}).$$

Lemma 1.9.3

Let I be a weak closed inner ideal in a JBW*-triple M and let (J_i) be a family of mutually orthogonal weak* closed ideals of M. Then*

$$I \cap \Big(\bigoplus J_i\Big) = \bigoplus I \cap J_i.$$

Proof

Let R denote the left hand side, above. Let $x \in R$ and choose $y \in R$ such that $x = \{yyy\}$ (see 1.7.2). We have $y = \sum x_i$, where $x_i \in J_i$ for all i. Therefore, since the x_i are mutually orthogonal

$$x = \sum \{yx_iy\} \in \bigoplus I \cap J_i,$$

proving that R is contained in $\bigoplus I \cap J_i$. The converse is clear. □

The orthogonal decomposition of weak* closed ideals in a JBW*-triple described by Theorem 1.9.2(e) is a starting point of an elaborate decomposition theory of 'types' of JBW*-triples [48], [49], [50].

A JBW*-triple M is said to be *type I* if each non-zero weak* closed ideal of M contains a non-zero abelian tripotent, and to be *continuous* if M contains no non-zero abelian tripotents. If M is a JBW*-triple and J is the smallest weak* closed ideal of M containing all abelian tripotents of M then

$$M = J \quad J^{\perp},$$

where J is type I and $J^{\perp}$ is continuous [48, (4.13)].

Let M be a JBW*-triple. If M has no non-trivial weak* closed ideals it is referred to as a *factor*. (When M is a JBW*-algebra this is equivalent to M having trivial centre.) Since non-zero abelian tripotents in a factor are minimal tripotents [48, (4.9)], the JBW*-triple factors of type I, also called Cartan factors (see Section 1.11 for details), are the factors that posses a minimal tripotent. The cardinality of a maximal orthogonal family of minimal tripotents in a type I factor M is an invariant referred to as the *rank* of M.

Let e and f be orthogonal tripotents of a JB*-triple A. Then $A_2(e)$ is orthogonal to $A_2(f)$. Indeed,

$$f \quad A_0(e).$$

Thus, the inner ideal $A_0(e)$ must contain the inner ideal generated by f. That is,

$$A_2(f) \quad A_0(e) = A_2(e)^{\perp}.$$

The following generalises this situation.

Lemma 1.9.4

Let a and b be elements of a JB-triple A, and let $r(a)$ and $r(b)$ be the corresponding range tripotents in A^{**}. The following are equivalent.*

(a) $a \quad b$.

(b) $A(a) \quad A(b)$.

(c) $A_2^{**}(r(a)) \quad A_2^{**}(r(b))$.

(d) $r(a) \quad r(b)$.

Proof

The implications (b)   (a), (c)   (d) are obvious and (d)

It follows that if a and b are orthogonal elements in a JB*-triple A, so that by Theorem 1.9.2(c) the JB*-subtriple

$$A(a) + A(b) = A(a) \quad {}_{\infty} A(b),$$

we have that

$$a + b = \max(a , b).$$

1.10 Support Tripotents

The key notion of support tripotents of functionals in the predual of a JBW*-triple, and much more, was studied in the seminal paper on JBW*-

In the next statement, (a) is a consequence of Theorem 1.8.3, Theorem 1.10.2 and 1.4.6(h), and (b) is due to Theorem 1.10.2 and the property that if J is a norm closed ideal of A and u is a minimal tripotent of A^{**} then $u \quad J^{**}$ or $u \quad (J^{**})^{\perp}$.

Corollary 1.10.4

Let A be a JB-triple. Let I be a norm closed inner ideal of A and let J be a norm closed ideal of A.*

(a) $\{\rho \quad \partial_e(A_1^*) : s(\rho) \quad I^{**}\} \longrightarrow \partial_e(I_1^*) \qquad (\rho \longrightarrow \rho|_I)$
is a bijection, the inverse of which is the norm one unique extension map.

(b) *Let $\rho \quad \partial_e(A_1^*)$. Then*

(i) $s(\rho) \quad J^{**}$ *if and only if* $\rho(J) = \{0\}$.

(ii) $s(\rho) \quad J^{**}$ *if and only if* $\rho(J) = \{0\}$.

1.10.5 Two functionals in the predual of a JBW*-triple are said to be *orthogonal* if their support tripotents are orthogonal.

Let $\rho, \tau \quad M_*$, where M is a JBW*-triple such that ρ and τ are orthogonal and

$$\rho \quad = \quad \tau \quad = 1.$$

Since

$$\rho = \rho \circ P_2(s(\rho)) \quad \text{and} \quad s(\rho) \quad s(\tau),$$

we have

$$\rho(s(\tau)) = 0 = \tau(s(\rho)).$$

Further, since $s(\rho) - s(\tau) \quad = 1$ and $(\rho - \tau)(s(\rho) - s(\tau)) = 2$, we have that

$$\rho - \tau \quad = 2.$$

Lemma 1.10.6

Let $\rho \in M_*$, where M is a JBW-triple. Suppose that ρ is a σ-convex sum*

$$\rho = \sum \lambda_n \rho_n \qquad \left(\sum \lambda_n = 1,\ \lambda_n > 0 \text{ for all } n\right)$$

of mutually orthogonal norm one functionals $\rho_n \in M_*$. Then

$$\|\rho\| = 1 \quad \text{and} \quad s(\rho) = \sum s(\rho_n).$$

Proof

Since $\rho_n(s(\rho_m)) = 0$ whenever $m \neq n$, we have

$$\rho\left(\sum s(\rho_n)\right) = \sum \lambda_n = 1.$$

Therefore, $s(\rho) \leq \sum s(\rho_n)$, by Theorem 1.10.1 (b). On the other hand

$$\sum \lambda_n \rho_n(s(\rho)) = 1,$$

giving $\rho_n(s(\rho)) = 1$ and hence $s(\rho_n) \leq s(\rho)$, for all n. Thus, since

$$\sum s(\rho_n) \leq s(\rho),$$

the required equality results. □

1.11 Cartan Factors

The Cartan factors (defined below) are exactly the type I JBW*-triple factors [49, 1.8]. There are six *generic types*. Namely, *rectangular*, *hermitian*, *symplectic* factors and *spin factors*, all four of which are JC*-triples, and two further *exceptional factors* $B_{1,2}$ and M_3^8.

In the following, complexifications of JB-algebras are assumed to be in

1.11.1

(1) $B(H, K)$ is the *rectangular* factor, $M_{n,m}$. We write $M_{n,n} = M_n$.

(2) For n □ 2, $\{x \quad B(H) : x^t = x\}$ is the *hermitian* factor S_n.

(3) For n □ 4, $\{x \quad B(H) : x^t = -x\}$ is the *symplectic* factor A_n.

Remarks

(a) If m □ n, so that K may be regarded as a closed subspace of H, we have

$$M_{m,n} \;=\; M_{n,m}$$

induced by $x \to x^t$. For finite m and n, $M_{m,n}$ is the space of $m \times n$ complex matrices.

(b) The hermitian factor S_n is a JBW*-algebra Jordan *-isomorphic to

$$B(H_{\mathrm{R}})_{sa} + iB(H_{\mathrm{R}})_{sa},$$

where H_{R} is the real Hilbert space such that

$H \;=\; H_{\mathrm{R}} + iH_{\mathrm{R}},$ given by $H_{\mathrm{R}} \;=\; \{h \quad H : jh = h\}$ [43, §7].

For 2 □ $n < \infty$, S_n 8(m)]TJF3411.955Tf13.1910T1Td002/

(4) *Spin Factors*

For any cardinal number n, the JBW*-triple *spin factor* V_n is the complexification

$$U_n + iU_n$$

of the JBW-algebra *real* spin factor [43, §6]

$$U_n = L_n \oplus \mathbb{R}1,$$

where L_n is a real Hilbert space of orthonormal dimension $n \geq 2$, with product and norm given by

$$(a + \alpha 1) \circ (b + \beta 1) = \beta a + \alpha b + (\langle a, b \rangle + \alpha\beta)1$$

and

$$\|a + \alpha 1\| = \|a\|_2 + |\alpha| \quad (\|\cdot\|_2 \text{ is the Hilbert norm}).$$

The spin factors V_n are reflexive and topologically equivalent to the complex Hilbert space of orthonormal dimension $n + 1$. By a *spin factor* V we shall always mean a *complex* spin factor V_n, for some n, unless otherwise explicitly stated.

The generic types of the two exceptional factors are next given. A clear account of these factors can be found in [36].

(5) $B_{1,2}$. This denotes the sixteen dimensional factor composed of the space of 1×2 matrices over the complex octonions.

(6) M_3^8. This is

1.11.2 Type I JBW*-algebra factors

The five kinds of Cartan factor M_n, S_n, A_{2n} (in the sense of (c)), V_n and M_3^8 are the *generic* types of JBW*-algebra factors of type I. There is some overlap with low dimensional spin factors. Namely,

$$V_2 = S_2, \quad V_3 = M_2, \quad \text{and} \quad V_5 =$$

In particular, (a) realises $B_{1,2}$ as a subtriple of M_3^8. We note that (b) realises $M_{m,n}$ as an inner ideal of $M_{s,t}$ (for $m \leq s$ and $n \leq t$) and (c) realises A_n as an inner ideal of A_{n+1}.

The smallest non-zero norm closed ideal of a Cartan factor is the *elementary ideal* of C, denoted by $K(C)$ [11].

Note: in this notation, standard in JB*-triple theory, if H is a Hilbert space then $K(H) = H$, and $K(B(H))$ is the C*-algebra of compact operators on H.

The following compilation is drawn from [11], [12], [21] and [41] and summarises salient properties of Cartan factors.

Theorem 1.11.4

(a) *Let C be a Cartan factor.*

  (i) *C is the weak* closed linear span of its minimal tripotents.*

  (ii) *$K(C)$ is the norm closed linear span of the minimal tripotents of C.*

  (iii) *$K(C)^{**} = C$, and $K(C) = C$ if and only if C has finite rank.*

  (iv) *Every weak* closed inner ideal of C is a Cartan factor.*

(b) *Let e be a minimal tripotent in a JBW*-triple M and let f be a minimal tripotent in a JB*-triple A.*

  (i) *The weak* closed ideal of M generated by e is a Cartan factor.*

  (ii) *The norm closed ideal of A generated by f is the elementary ideal of a Cartan factor.*

1.12 Atomic JBW*-Triples and Decomposition

A JBW*-triple is defined to be *atomic* if it is the weak* closed linear span of its minimal tripotents.

In Theorem 1.12.1, part (a) follows from Theorem 1.11.4 (b)(i), and part (b) is [41, Remark 2.8].

Theorem 1.12.1

Let M be an atomic JBW-triple.*

(a) *M is an ℓ_∞-sum of Cartan factors.*

(b) *Every norm one functional in the predual of M is a σ-convex sum of mutually orthogonal functionals in $\partial_e(M_{*,1})$.*

If M is a JBW*-triple we denote the smallest weak* closed ideal of M containing the minimal tripotents of M by M_{at} and shall refer to it as the *atomic part* of M. By Theorem 1.9.2(e), the orthogonal decomposition of weak* closed ideals

$$M = M_{at} \oplus (M_{at})^{\perp}$$

is automatic, and $(M_{at})^{\perp}$ contains no minimal tripotents. In light of these remarks the following is crucial.

Theorem 1.12.2 [41, Theorem 2]

If M is a JBW-triple then M_{at} is the weak* closed linear span of the minimal tripotents of M. Hence, M_{at} is an atomic JBW*-triple.*

Let A be a JB*-triple and let $\rho \in A^{*}$. Then ρ is defined to be an *atomic functional* of A if $s(\rho) \in A^{**}_{at}$. It is immediate from Theorem 1.10.2(b) that ρ is atomic for all $\rho \in \partial_e(A^{*}_{1})$. Consider the orthogonal decomposition

$$A^{**} = A^{**}_{at} \oplus (A^{**}_{at})^{\perp}.$$

Let

$$P_{at} : A^{**} \longrightarrow A^{**}_{at}$$

denote the canonical projection. Via Theorem 1.10.1 we have that the set S of atomic functionals of A is given by

$$S = \{\rho \in A^* : \rho((A^{**}_{at})^{\perp}) = \{0\}\} = \{\rho \in A^* : \rho = \rho \circ P_{at}\}.$$

Chapter 2

Hahn-Banach Extensions

2.1 Introduction

Let E be a norm closed subspace of a Banach space X and let $\rho \in E^*$. The Hahn-Banach theorem guarantees the existence of a norm preserving extension of ρ, also referred to as a Hahn-Banach extension of ρ, in X^*. Such extensions need not be unique. Moreover, if ρ has two distinct norm preserving extensions ϕ and ψ in X^*, then it has uncountably many, since each element of the straight line joining ϕ to ψ in X^*,

$$\{(1-\alpha)\phi + \alpha\psi : \alpha \in [0,1]\},$$

is again a norm preserving extension of ρ.

The question of existence of unique Hahn-Banach extensions seems to be one of considerable complexity, as may be seen by the recent article [5] and references therein. The subject impinges upon JB*-triple theory via a path pioneered in [30], [33], and [35] (see Theorems 1.8.3 and 1.8.6). Involved, broadly speaking, are generalisations of M-ideal theory. Extensive accounts of L- and M-theory can be found in [2], [3], [7] and [44] from which main definitions and some results are recalled. Pertinent Banach space 'smoothness' properties are briefly reviewed and applied in the context of JB*-triples. In addition, useful conclusions are drawn in connection with unique Hahn-Banach extensions of dual ball extreme points.

2.1.1 We shall work with the following definitions. Let X be a Banach space and let E be a norm closed subspace of X.

(a) E is said to have the *extension property* in X if each $\rho \in E^*$ has unique norm preserving extension in X^*.

(b) E is said to have the *extreme extension property* in X if each $\rho \in \partial_e(E_1^*)$ has unique extension in $\partial_e(X_1^*)$.

(c) X is said to have the *extension property* if every norm closed subspace of X has the extension property in X.

(d) X is said to have the *extreme extension property* if every norm closed subspace of X has the extreme extension property in X.

2.2 L-Summands, M-Summands and Ideals

2.2.1 Let E be a closed subspace of a Banach space X. We have the following definitions.

(a) E is an *L-summand* of X if there exists a closed subspace F of X such that $E \oplus F = X$ and

$$\|x + y\| = \|x\| + \|y\|$$

for each $x \in E$ and $y \in F$.

(b) E is an *M-summand* of X if there exists a closed subspace F of X such that $E \oplus F = X$ and

$$\|x + y\| = \max\{\|x\|, \|y\|\}.$$

for each $x \in E$ and $y \in F$.

We also have the following related definitions, where P is a projection on X.

(c) P is said to be an *L-projection* if

$$x \;=\; Px \;+\; x - Px$$

for each $x \quad X$.

(d) P is said to be an *M-projection* if

$$x \;=\; \max\{\; Px \;,\; x - Px \;\}$$

for each $x \quad X$.

(e) P is said to be *neutral* if P is contractive, and if $x \quad X$ such that $Px \;=\; x$, then $Px = x$.

The complement $I - P$ of an M-projection P is an M-projection. Similarly, complements of L-projections are L-projections. Every L-projection is a neutral projection. The map $P \to P(X)$ defines bijections from the set of

(i) L-projections on X onto the set of L-summands of X;

(ii) M-projections on X onto the set of M-summands of X.

We further recall that

(iii) $P : X \to X$ is an M-projection if and only if $P^* : X^* \to X^*$ is an L-projection;

(iv) $P : X \to X$ is an L-projection if and only if $P^* : X^* \to X^*$ is an M-projection.

2.2.2 Let E be a norm closed subspace of a Banach space X. The topological annihilator of E in X^* is denoted by

$$E^\circ \;=\; \{\rho \quad X^* : \rho(E) = \{0\}\}.$$

The *subset* E (it need not be a linear subspace) of X^* given by

$$E \;=\; \{\rho \quad X^* : \; \rho|_E \;=\; \rho \;\}$$

satisfies

$$E^\circ \cap E \;=\; \{0\} \quad \text{and} \quad X^* = E^\circ + E \qquad \text{(set addition)}.$$

(a) E is said to be an *M-ideal* of X if E° is an L-summand of X^*. In which case, E is a subspace of X^* and is the complementary L-summand of E° in

$$X^* \;=\; E^\circ \quad E\,.$$

(b) E is said to be an *N-ideal* of X if E is a subspace of X^*. In which case

$$X^* \;=\; E^\circ \quad E$$

and the projection onto E is contractive and neutral [35].

(c) E is said to be a *Banach ideal* of X if there is a contractive projection P on X^* with

$$\ker P \;=\; E^\circ.$$

Remarks. The term *Banach ideal* used above is non-standard. Subspaces satisfying (c) are referred to simply as *ideals* of X in [59], for example. But the latter usage conflicts widely with that used elsewhere in this thesis. The term N-ideal was used in [35].

2.2.3 Consider now a norm closed subspace E of a Banach space X. Suppose there is a contractive projection P on X^* such that

$$\ker P = E^\circ.$$

Let $\rho \quad X^*$. By definition, ρ and $P\rho$ agree on E and for each $\tau \quad E^\circ$ we have

$$P\rho = P(\rho - \tau) \quad \square \quad \rho - \tau$$

so that

$$P\rho \quad \square \quad \rho + E^\circ = \rho|_E$$

and thus $P\rho = \rho|_E$. Hence, $P\rho$ is a Hahn-Banach extension of $\rho|_E$ and $P(X^*)$ is contained in E .

Now suppose that, in addition, E has the extension property in X. Then $P\rho$ must be the unique norm preserving extension of $\rho|_E$, and $P(X^*$ B2. Hence, We note that if E is an N-ideal of a Banach space X and

$$\phi : E^* \longrightarrow X^*$$

denotes the norm preserving unique extension map then

$$\phi : E^* \longrightarrow E \qquad \text{and} \qquad \sigma : E \longrightarrow E^* \qquad (\rho \longrightarrow \rho|_E)$$

are mutually inverse surjective linear isometries.

The relevance of the above for JB*-triples is that the M-ideals of a JB*-triple are precisely its norm closed ideals [6, Theorem 3.2] and the following variation of Theorem 1.8.3.

Theorem 2.2.5 [30], [33], [35]
The following are equivalent for a JB-subtriple E of a JB*-triple X.*

(a) *E has the extension property in X.*

(b) *E is an N-ideal of X.*

(c) *E is an inner ideal of X.*

2.3 The Extension Property

2.3.1 We recall (for example, s955Tf-F357.57.714.346Tf-13511.9007.71TJF3Td[(1px7y1c1e-

Each of the conditions (a) and (b) passes to all norm closed subspaces (via the Hahn-Banach theorem in the case of (a)).

There is a well-known (partial) duality between smoothness and strict convexity due to Klee.

Theorem 2.3.2 [54, A1.1]

Let X be a Banach space.

(a) *If X^* is smooth then X is strictly convex.*

(b) *If X^* is strictly convex then X is smooth.*

(c) *If X is reflexive then the converses of* (a) *and* (b) *are true.*

2.3.3 All Hilbert spaces are both smooth and strictly convex. It is clear that neither $C \;{}_{\infty} C$ nor $(C \;{}_{\infty} C)^* = C \;{}_{1} C$ is strictly convex and thus, by Theorem 2.3.2(c), neither of these spaces is smooth. Hence, if X is smooth or strictly convex then X cannot contain an isometric copy of $C \;{}_{\infty} C$.

The connection with the extreme extension property comes from the following theorem.

Theorem 2.3.4 [37], [68, Theorem 6]

A Banach space X has the extension property if and only if X^* is strictly convex.

The proceeding elementary result is useful.

Lemma 2.3.5

The following are equivalent for a JB-triple A.*

(a) *Every norm one element is a tripotent.*

(b) *No two non-zero elements are orthogonal.*

(c) *A is a Hilbert space.*

Proof

To see that (a) (b), note that if u and v are non-zero orthogonal tripotents of A then $u + \frac{1}{2}v$

(b) *If* $\rho \quad \partial_e(E_1^*)$ *then*

(i) $E(\rho, E)$ *is a face of* X_1^*;

(ii) ρ *has an extension in* $\partial_e(X_1^*)$.

Proof

(a) Note that $E(\rho, E)$ is weak* closed, since if ϕ is the weak* limit of a net (ϕ_λ) in $E(\rho, E)$, then

$$\phi(a) \; = \; \lim \phi_\lambda(a) \; = \; \rho(a)$$

for all a in E, so that $\phi \quad E(\rho, E)$. Now, since X_1^* is weak* compact, the same is true of $E(\rho, E)$.

For convexity, let $\lambda \quad [0, 1]$ and $\phi, \psi \quad E(\rho, E)$. Then

$$\lambda\phi + (1 - \lambda)\psi \quad X_1^*$$

and

$$(\lambda\phi + (1 - \lambda)\psi)|_E \; = \; \lambda\phi|_E + (1 - \lambda)\psi|_E \; = \; \rho$$

so that $\lambda\phi + (1 - \lambda)\psi \quad E(\rho, E)$, as required.

(b)(i) Suppose $\lambda \quad (0, 1)$ and $\phi, \psi \quad X_1^*$ such that

$$\lambda\phi + (1 - \lambda)\psi \quad E(\rho, E).$$

Since this functional agrees with ρ on E,

$$\lambda\phi|_E + (1 - \lambda)\psi|_E \; = \; \rho$$

so that $\phi|_E = \psi|_E = \rho$, because ρ is an extreme point of E_1^*. Therefore ϕ and ψ lie in $E(\rho, E)$, as required.

(ii)

Corollary 2.4.2

Let E be a norm closed subspace of a Banach space X, and let $\rho \quad \partial_e(E_1^*)$.

(a) *ρ has a unique extension $\overline{\rho}$ in $\partial_e(X_1^*)$ if and only if $\overline{\rho}$ is the unique extension of ρ in X_1^*.*

(b) *E has the extreme extension property in X if and only if each $\rho \quad \partial_e(E_1^*)$ has unique extension in X_1^*.*

Proof

(a) If ρ has unique extension $\overline{\rho}$ in $\partial_e(X_1^*)$, then $\partial_e(E(\rho, E)) = \{\overline{\rho}\}$ by Lemma 2.4.1 and hence $E(\rho, E) = \{\overline{\rho}\}$ by the Krein-Milman theorem. Conversely, $E(\rho, E) = \{\overline{\rho}\}$ implies $\overline{\rho} \quad \partial_e(X_1^*)$, again by Lemma 2.4.1.

(b) This is immediate from (a). □

2.4.3 Let x be a norm one element of a Banach space X. Even more straightforwardly than in Lemma 2.4.1, it is seen that E^x is a weak* compact convex subset of X_1^* and, moreover, is always a face of X_1^*. Therefore,

(a) $\partial_e(E^x) \quad \partial_e(X_1^*)$.

There will be a unique $\phi \quad X_1^*$ attaining its norm at x if and only if $E^x = \{\phi\}$. Since E^x is a face of X_1^*, this is equivalent to saying that $\phi \quad \partial_e(X_1^*)$, by the Krein-Milman theorem. It follows that

(b) X is smooth if and only if $N(X_1^*) \quad \partial_e(X_1^*)$.

We have the following characterisation of smoothness in terms of unique extension conditions.

Proof

If X has the extreme extension property then it is certainly smooth, by Proposition 2.4.4, so that

$$N(X_1^*) \quad \partial_e(X_1^*),$$

by 2.4.3(b). If, in addition, $\partial_e(X_1^*)$ is norm closed then we deduce from the Bishop-Phelps theorem ($N(X_1^*)$ is norm dense in $S(X_1^*)$) that

$$\partial_e(X_1^*) = S(X_1^*),$$

so that X^* is strictly convex which, using Theorem 2.3.4, completes the proof. □

2.4.6 We remark that $\partial_e(X_1^*)$ is norm closed whenever X is a JB*-triple [16, Proposition 4]. Without appealing to this fact it is in any case immediate from Theorem 2.3.6 and Proposition 2.4.4 that the extension property and the extreme extension property coincide for JB*-triples.

We shall close this section with three observations used later.

Proposition 2.4.7 [18]

Let X be a Banach space and let E be a closed subspace of X. Suppose that E has the extreme extension property in X. Then the unique extension map

$$\begin{array}{ccc} \partial_e(E_1^*) & \longrightarrow & \partial_e(X_1^*) \\ \rho & \longrightarrow & \bar{\rho} \end{array}$$

;

Proposition 2.4.8

Let X be a Banach space and let E be a closed subspace of X. Suppose that E has the extreme extension property in X and there exists a contractive projection, Q, from X onto E. Then for each $\rho \quad \partial_e(E_1^*)$, $Q^*(\rho)$ is the unique extension of ρ in $\partial_e(X_1^*)$. Moreover, Q is the unique contractive projection from X onto E.

Proof

Let $\rho \quad \partial_e(E_1^*)$. Since Q is the identity function on E, $Q^*(\rho)$ extends ρ, and $Q^*(\rho) \;=\; \rho$ since Q^* is contractive. So, $Q^*(\rho)$ is the unique extension of ρ in X_1^*, by assumption.

To prove uniqueness, suppose that there exists another contractive projection P from X onto E. Then P^* and Q^* agree on $\partial_e(E_1^*)$. But P^* and Q^* are weak* continuous, and so also agree on E_1^*, by the Krein-Milman theorem. Hence, $P^* = Q^*$ and consequently $P = Q$. □

Corollary 2.4.9

Let I be a norm closed inner ideal of a JB-triple A.*

(a) *There is at most one contractive projection from A onto I.*

(b) *If $P : A \to A$ is a contractive projection such that*

$$P(A) = I,$$

then P is a structural projection.

Proof

(a) This follows from Theorem 2.2.5 and Proposition 2.4.8.

(b) By Theorem 1.8.6, there is a structural projection

$$Q : A^{**} \longrightarrow I^{**},$$

and since I^{**} must have the extension property in A^{**}, being a weak* closed and thus norm closed inner ideal of A^{**}, we have

$$Q = P^{**},$$

by Proposition 2.4.8. Hence, P is a structural projection. □

2.5 c_0-sums and M-Orthogonality

2.5.1 Let $(E_i)_{i\in I}$ be a family of Banach spaces. We shall write

(a) $(\square\, E_i)_\infty = \{\square\, x_i : (\ x_i\ )\quad {}_\infty(I)\}$

and

(b) $(\square\, E_i)_0 = \{\square\, x_i : (\ x_i\ )\quad c_0(I)\}$

to express the respective $\ _\infty$ and c_0-sums of the E_i as i ranges over an indexing set I.

In this way, each E_j is an M-summand of $(\square\, E_i)_0$ (and $(\square\, E_i)_\infty$) and we have the natural linear isometry

(c) $E_j = E_j^* \quad (\rho \to \rho|_{E_j})$.

Similarly, we use

(d) $(\square\, E_i)\ _1 = \{\square\, x_i : (\ x_i\ )\quad {}_1(I)\}$

for the $\ _1$-sum of the E_i. In this case, each E_i is an L-summand of the $\ _1$-sum.

We have the further linear isometries

(e) $((\square\, E_i)\ _1)^* = (\square\, E_i^*)_\infty \quad (\rho \to \square\, \rho|_{E_i})$;

as

$$((\square\, E_i)_0)^* = (\square\, E_i^*)\ _1 \quad (\rho \to \square\, \rho|_{E_i}),$$

the latter giving the equality)

2.5.2 Let $X = (\square\, E_i)_1$, where $(E_i)_{i \in I}$

Proposition 2.5.4

A c_0-sum of two or more non-zero M-orthogonal JB-subtriples of a JBW*-triple factor M cannot have the extension property in M.*

Proof

Let (E_i) be a family of non-zero M-orthogonal JB*-subtriples of a JBW*-triple factor M such that

$$\left(\bigoplus E_i\right)_0$$

has the extension property in M. Fix i_0. Then

$$E \oplus F = \left(\bigoplus E_i\right)_0,$$

where $E = E_{i_0}$ and F is the c_0-sum of the remainder.

Put $A = E \oplus F$. Since it has the extension property in M, A must be a norm closed inner ideal of M, by Theorem 2.2.5. In particular, A is a JB*-triple with orthogonal ideals E and F, since each is an M-ideal of A. Thus,

$$\overline{A}^w = \overline{E}^w \oplus_\infty \overline{F}^w$$

is a weak* closed inner ideal of M with orthogonal non-zero weak* closed ideals $\overline{E}^w$ and $\overline{F}^w$ and so is not a factor. Since every weak* closed inner ideal in a JBW*-triple factor is again a factor, this is a contradiction. □

As consequences we note that given a family $(E_i)_{i\in I}$ of mutually M-orthogonal norm closed subspaces of a Banach space Y,

(a) if each E_i is a E $)_i$

Proposition 2.5.5

Let $(E_i)_{i\in I}$ be a family of mutually M-orthogonal norm closed subspaces of a Banach space X. Let E_i

Chapter 3

Cartan Factors and Unique Predual Extensions

3.1 Introduction

Given a weak* closed subtriple M of a JBW*-triple N, Theorem 1.8.3 shows that every functional in the predual of M has a unique norm preserving extension in the predual of N if and only if M is an inner ideal of N. In consistency with the general theme of our thesis, in this chapter we shall investigate the consequences of the existence of unique norm preserving extensions in N_* of single functionals in $\partial_e(M_{*,1})$ and of other atomic functionals.

As shall be seen, without any essential loss of information, our investigation quickly reduces to the case where both M and N are Cartan factors. Amongst other things, in this case we show that the existence of one functional in $\partial_e(M_{*,1})$ with unique norm one extension in N_* compels the same condition upon all functionals in $\partial_e(M_{*,1})$. However, this particular unique extension condition does not imply that M is an inner ideal of N because of obstructions involving generic type. Indeed, in these circumstances, it is the case that more often than not the Cartan factors M and N in question have different generic type.

Inspection of finite dimensional examples reveals algebraic and geometric obstructions alluded to above. For instance, given finite $n \geq 2$, the triple embeddings

$$\alpha_1 : S_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C}) \quad \text{and} \quad \alpha_2 : M_n(\mathbb{C}) \longrightarrow A_{2n}(\mathbb{C}),$$

where α_1 is inclusion and α_2 maps x to

$$\begin{pmatrix} 0 & x^T \\ -x & 0 \end{pmatrix}$$

both implement the extreme extension property, as shall be seen in the coming general investigation, as therefore does their composition. However, neither image is an inner ideal in its codomain. The validity of these observations remain unaffected by further (appropriate) composition with the triple embeddings

$$M_{m,n} \longrightarrow M_n, \quad M_n \longrightarrow M_{n,l}\ (m \le n \le l) \quad \text{and} \quad A_{2n} \longrightarrow A_{2n+1},$$

obtained by suitable addition of 0's (see 1.11.3). On the other hand, the triple embeddings

$$\beta_1 : M_n(\mathbb{C}) \longrightarrow S_{2n}(\mathbb{C}) \quad \text{and} \quad \beta_2 : A_n(\mathbb{C}) \longrightarrow M_{2n}(\mathbb{C}),$$

where β_1 maps x to

$$\begin{pmatrix} 0 & x^T \\ x & 0 \end{pmatrix}$$

and β_2 is inclusion, do not implement the extreme extension property.

It turns out that these observations are characteristic of some general rules. Suppose that M and N are hermitian, rectangular or symplectic Cartan factors and that M is a Cartan subfactor of N for which there exists ρ in $\partial_e(M_{*,1})$ with unique norm one extension in N_*. Then, as shall be proved in the sequel, in terms of generic type we have the following table of the possible generic types of N relative to the generic type of M.

extension in N_*

Corollary 3.2.2

Let I be a weak closed inner ideal of a JBW*-triple M and let*

$$I_* \longrightarrow M_* \qquad (\tau \longrightarrow \overline{\tau})$$

denote the norm preserving unique extension map. Then

$$\{\overline{\tau} \quad M_* : \tau \quad I_*\} = \{\rho \quad M_* : s(\rho) \quad I\} = I \ \cap M_*$$

and

$$I \ \cap M_* \longrightarrow I_* \qquad (\rho \to \rho|_I)$$

is a surjective linear isometry.

Proof

The first equality of the statement follows from Lemma 3.2.1 and the fact that given a norm one element ρ in M_* such that $s(\rho)$ lies in I, then $\rho|_I$

(a) *If* $\rho|_I = \phi|_I$, *then* $\rho = \phi|_M$.

(b) ϕ *is the unique norm preserving extension of* ρ *in* N_* *if and only if* ϕ *is the unique norm preserving extension of* $\rho|_I$ *in* N_*.

Proof

(a) Put $u = s(\rho)$, let $\|\rho\| = 1$ and suppose that

$$\rho|_I = \phi|_I.$$

Since $\phi(u) = 1$, then Theorem 1.10.1(b) shows that

$$\phi(x) = \phi(P_2(u)(x)),$$

for all $x \in N$. Therefore, since $P_2(u)(M) \subseteq I$, for each x in M we have

$$\phi(x) = \rho(P_2(u)(x)) = \rho(x),$$

where the second equality again derives from Theorem 1.10.1.

(b) Suppose that ϕ is the unique norm preserving extension of ρ in N_* and let ψ be any extension of $\rho|_I$ with

$$\|\psi\| = \|\rho|_I\| \qquad (= \|\rho\|).$$

Since $\psi|_I = \rho|_I = \phi|_I$, (a) implies that

$$\psi|_M = \rho = \phi|_M,$$

so that $\psi = \phi$, by assumption. The converse is clear. □

3.2.5 Let us finally note that if C is a Cartan factor with corresponding elementary ideal $K(C)$, then, since $K(C)^{**} = C$ (Theorem 1.11.4 (a)(iii)), we have $K(C)^* = C_*$, so that upon identifying C_* with the L-summand $K(C)$ of C^* we have

$$C^* = C_* \oplus_1 K(C)^\circ \qquad (\ell_1\text{-sum}).$$

Consequently,

$$\partial_e(C_1^*) \;=\; \partial_e(C_{*,1}) \qquad \partial_e((K(C)^\circ)_1) \quad \text{(disjoint union)}$$

and the functionals in $\partial_e(C_{*,1})$ are precisely those in $\partial_e(C_1^*)$ that do not vanish on $K(C)$.

3.3 Unique Extreme Predual Extensions

If $\rho \quad \partial_e(M_{*,1})$, where M is a JBW*-triple and C is the weak* closed ideal of M generated by $s(\rho)$, then, by Theorem 1.10.1 together with [21], C is a Cartan factor and ρ vanishes on the orthogonal complement of C in M. Thus, ρ 'lives' on C and we may regard ρ as a member of $\partial_e(C_{*,1})$. In this sense nothing is lost if it is assumed that M coincides with the Cartan factor C when discussing consequences of ρ having a unique norm preserving extension in the predual of a containing JBW*-triple.

We begin by studying unique norm one weak* continuous extensions from the extreme predual of a Cartan subfactor of a JBW*-triple. Since the norm closed predual ball of a JBW*-triple need not be weak* closed, the Krein-Milman theorem is not directly applicable. Essential use is made throughout of Theorem 1.10.2, the bijective correspondence between predual ball extreme points of a JBW*-triple and minimal tripotents. Such is its ubiquity we sometimes employ it tacitly.

Lemma 3.3.1

Let C be a JBW-subtriple of a JBW*-triple M, where C is a Cartan factor. Let $\rho \quad \partial_e(C_{*,1})$ such that ρ has a unique norm one extension $\bar{\rho}$ in M_*. Then $\bar{\rho} \quad \partial_e(M_{*,1})$, $s(\rho) = s(\bar{\rho})$ and $s(\rho)$ is a minimal tripotent of M.*

Proof

Suppose that $\bar{\rho}$ is the unique extension of ρ in $M_{*,1}$, and let $u = s(\rho)$. We shall show that the JBW*-algebra $M_2(u)$ has only one normal state.

Let ϕ be a normal state of $M_2(u)$ and let $\psi = \phi \circ P_2(u)$. Then $\psi \quad M_{*,1}$ and $\psi(u) = 1$. Therefore $\psi|_C$ belongs to $C_{*,1}$ and is equal to ρ, by Theorem 1.10.2. Hence, by the above assumption, $\psi = \bar{\rho}$. So, on $M_2(u)$, $\phi = \bar{\rho}$. Therefore ϕ is the unique normal state of $M_2(u)$ and so the latter must be of dimension one. Hence $M_2(u) = \mathbb{C}u$, giving that u is a minimal tripotent of M and, since $\bar{\rho}$

Proof

Let $\phi \quad \partial_e(C_{*,1})$. Using Theorem 1.8.4, let τ be any extension of ϕ in $M_{*,1}$. By Lemma 3.3.1 and Lemma 3.3.2 we have that $s(\phi)$ is minimal in M. Since $\tau(s(\phi)) = 1$ we have that, using Theorem 1.10.2, $\tau \quad \partial_e(M_{*,1})$ and must be the unique extension of ϕ in $M_{*,1}$.

where the $\tau_n \; \partial_e(D_{*,1})$. But then ρ is the σ-convex sum

$$\rho = \sum \lambda_n \left(\tau_n|_C\right).$$

Therefore, since $\rho \; \partial_e(C_{*,1})$, the sum is degenerate and all the $\tau_n|_C$ equal ρ. Hence, by the assumption, all the τ_n equal $\bar{\rho}$. Hence, $\phi = \bar{\rho}$.

(d) (c) This follows from the above proof of (b) (a).

This shows that (a), (b), (c), (d), (e) and (f) are equivalent. The condition (f) implies that $K(C) \; K(D)$ and the condition (d) now implies that each element of $\partial_e(K(C)^*_1)$ has unique extension in $\partial_e(K(D)^*_1)$, so that $K(C)$ has the extreme extension property in $K(D)$ and hence in D, giving (g).

(g) (c) Assume (g). Then each $\rho \; \partial_e(C_{*,1})$ has unique extension in D^*_1 and hence in $D_{*,1}$ (since each such ρ has an extension in $D_{*,1}$). This completes the proof. □

We shall proceed to develop an analysis of Cartan subfactors of Cartan factors.

Proposition 3.3.5

Let C be a Cartan factor JBW-subtriple of a Cartan factor D. Let u be a tripotent of C such that rank.9Tf2.320Td[(1F3111.9Tf29Tf21-114.6193043)]TJF111.9Tf6.08Td[(.)]TJE*

for all i. It follows that there exists km orthogonal minimal projections in N with sum 1. Hence, $n = km$. □

Proposition 3.3.7

Let C be a Cartan factor JBW-subtriple of a Cartan factor D such that*

$$\operatorname{rank}(D) < 2\operatorname{rank}(C) < \infty.$$

Then C has the extreme extension property in D. In particular, this holds when

$$\operatorname{rank}(C) = \operatorname{rank}(D) < \infty.$$

Proof

Suppose that $\operatorname{rank}(C) = n$

(a) *C and D are spin factors,*

(b) *$C = B_{1,2}$ and $D = M_3^8$,*

(c) *rank$(C) = 2$ and rank$(D) = 2$ or 3.*

Proof

Since all spin factors have rank 2 and rank$(B_{1,2}) = 2$, and rank$(M_3^8) = 3$, the statements (a) and (b) are immediate from the statement (c) which, in turn, is immediate from Proposition 3.3.7. □

The *classification scheme* of Dang and Friedman [21, p305] shows that the generic type of a Cartan factor C of rank greater than one is completely determined up to linear isometry by the (spin factor) structure of the Peirce 2-space $C_2(u)$ associated with a rank 2 tripotent u of C according to the following table

3.3.9

C	V_n	S_n	$M_{n,k}$	$A_n, n \geq 4$	$B_{1,2}$	M_3^8

(i) *For prescribed C, the (only) possible generic type of D is as follows.*

(b) The table follows from Proposition 3.3.7 and that $M_{n,m}$, S_n, A_{2n} and A_{2n+1} all have rank n.

For example, let $n \leq m < 2n < \infty$ and let $C = S_n$ and $D = S_m$. Then we have

$$\operatorname{rank}(D) = m < 2n = 2\operatorname{rank}(C) < \infty$$

so that C has the extreme extension property in D by Proposition 3.3.7. The rest of the table follows similarly.

(c) We prove the second row of the first table (the proof of the other two rows is almost identical). The second table follows immediately from the first table.

Let u be a rank 2 tripotent of C and suppose that C is rectangular. Then, by Proposition 3.3.5 together with table 3.3.9,

$$C_2(u) = V_3$$

and $C_2(u)$ has the extreme extension property in $D_2(u)$. Hence, we must have that

$$D_2(u) = V_3 \quad \text{or} \quad D_2(u) = V_5$$

and it follows, again by table 3.3.9, that D is either rectangular or symplectic. □

3.4 Inner Ideals

3.4.1 In this section we shall consider the question of when the 'single' unique extension condition from $\partial_e(C_{*,1})$ to $\partial_e(D_{*,1})$, introduced and discussed in the previous section, forces C to be an inner ideal of D, where C and D are Cartan factors and C is a JBW*-subtriple of D. When C has rank 1 (that is, is a Hilbert space) this question is easily answered by use of the result of Dang and Friedman [21, p306] that the closed subspace generated by a family of mutually collinear minimal tripotents in a JBW*-triple M is a Hilbert space and an inner ideal of M. In order to be clear we shall formally record this result.

Theorem 3.4.2 [21, p306]
Let M be a JBW-triple and let (e_α) be a family of mutually collinear minimal tripotents of M. Then the closed linear span of the (e_α) is linearly isometric to a Hilbert space and is a weak* closed inner ideal of M.*

Conversely, every weak closed inner ideal of M linearly isometric to a Hilbert space arises in this way.*

Corollary 3.4.3
If C is a Hilbert space JBW-subtriple of a Cartan factor D and there exists $\rho \quad \partial_e(C_{*,1})$ with unique extension in $\partial_e(D_{*,1})$, then C is an inner ideal of D.*

Proof
This follows from Theorem 3.3.4 (b)$\Leftrightarrow$(f) and Theorem 3.4.2. □

Let v be a minimal tripotent in a Cartan factor C. By the *classification scheme* of [21, p305] (see A: Case 1, Case 2), the Peirce 1 space $C_1(v)$ is a Hilbert space if and only if C is a Hilbert space or is hermitian and in that case C contains a tripotent u such that $u \quad C_1(v)$ and $v \quad C_1(u)$ (that is, u, v are collinear) only when C is a Hilbert space. In particular, an hermitian Cartan factor cannot contain two collinear minimal tripotents. The following lemma is a restatement of this property.

Lemma 3.4.4
If H is a Hilbert space and an inner ideal of an hermitian Cartan factor, then $\dim(H) = 1$.

In 3.4.5 and 3.4.6 below C and D are Cartan factors where C is a JBW*-subtriple of D.

3.4.5 *Spin Factors and Exceptional Factors*

Let D be a spin factor or an exceptional factor. By Proposition 3.3.7 and Corollary 3.3.8, all rank 2 subfactors of D satisfy the extreme extension property in D, as do all rank 3 subfactors when $D = M_3^8$.

(a) If D is a spin factor, then all proper inner ideals are of rank 1, by [34, Lemma 5.5].

If D

(We are allowing C or $D = V_2, V_3$ or V_5.) If C is an inner ideal of D then for any rank 2 tripotent u of C we have

$$C_2(u) = D_2(u) = V_2, V_3 \text{ or } V_5$$

so that C and D have the same generic type (see table 3.3.9). Our present intention is to prove, conversely, that if C and D have the same generic type and there exists ρ $\partial_e(C_{*,1})$ with unique extension in $\partial_e(D_{*,1})$, then C must be an inner ideal of D.

Lemma 3.4.7

Let M be a JBW-subtriple of a JBW*-triple N such that $M_2(u)$ is an inner ideal of N for each complete tripotent of M. Then M is an inner ideal of N.*

Proof

Let x M. We must show that $\{xNx\}$ M. By the triple functional calculus, x lies in $M_2(r(x))$ (Theorem 1.7.1). In turn, via [49, 3.12], choose a complete tripotent u of M such that $r(x)$ lies in $M_2(u)$. This gives

$$x \quad M_2(r(x)) \quad \text{[illegible]}$$

a c[illegible]is t pr

Theorem 3.4.8

Let C and D be hermitian, rectangular or symplectic Cartan factors of rank □ *2 and of the same generic type, where C is a JBW*-subtriple of D. Then there exists ρ in $\partial_e(C_{*,1})$ with unique extension in $\partial_e(D_{*,1})$ if and only if C is an inner ideal of D. In which case, every element of C^* has a*

Let $x \quad D_2(u)_{sa}$. Then x is the weak* limit of $\{e_F x e_F\}$. But, for each F, e_F has finite rank, so that

$$e_F x e_F \quad D_2(e_F) = C_2(e_F) \quad C_2(u).$$

Therefore, $x \quad C_2(u)$. Hence, $C_2(u) = D_2(u)$ which, by Lemma 3.4.7, proves that C is an inner ideal of D. The converse and final statement is an application of Theorem 1.8.3. □

We shall proceed to derive a number of corollaries.

Corollary 3.4.9

Let C and D be Cartan factors, where C is a JBW-subtriple of D. If there exists $\rho \quad \partial_e(C_{*,1})$ with unique extension in $\partial_e(D_{*,1})$ and C is symplectic or D is hermitian, then C is an inner ideal of D.*

Proof

If C is a symplectic Cartan factor then, by Theorem 3.3.11(c), D must also be symplectic. Similarly, C must be hermitian if D is hermitian. In which that $C \quad D$ and we may suppose that $\rho \quad \partial_e(C_{*,1})$ with unique extension in $\partial_e(D_{*,1})$. Hence, C is an inner ideal of D, by Corollary 3.4.9, and therefore C is an inner ideal of N. □

Corollary 3.4.11

Let C and D be Cartan factors, where C is a JBW-subtriple of D. If C and D posses a common unitary tripotent such that*

(a) *C and D are hermitian, rectangular or symplectic and of the same generic type and,*

(b) *there exists $\rho \quad \partial_e(C_{*,1})$ with unique extension in $\partial_e(D_{*,1})$,*

then $C = D$.

Proof

In this case C is an inner ideal of D, by Theorem 3.4.8, and C contains a unitary tripotent u of D, so that $C = C_2(u) = D_2(u) = D$. □

Our final corollary of this section does not make exp3f4.7331.79cner idT100-35897-3d[(and *possess a common*

Proof

In each case, by Theorem 3.3.11, C has the extreme extension property in D, and C and D are of the same generic type. Hence, Theorem 3.4.8 applies to give the desired conclusions. □

Proof

In order to obtain a contradiction, assume that U is properly contained in U'. We have

$$U = L \oplus \mathbb{R}1 \quad \text{and} \quad U' = L' \oplus \mathbb{R}1$$

where L and L' are real Hilbert spaces with L being a proper closed subspace of L'. In particular, the Hilbert space orthogonal complement of L in L' is non-zero.

By 3.5.2, we have

$$\rho = \rho_h^U, \text{ for some } h \in L \text{ with } \|h\| < 1.$$

Choose any non-zero element h' in L' orthogonal to L satisfying

$$\|h'\| < (1 - \|h\|^2)^{1/2}$$

so that

$$\|h + h'\|^2 = \|h\|^2 + \|h'\|^2 < 1.$$

Now, for any such h', of which there are infinitely many, we have

$$\rho_{h+h'}^U(\alpha 1 + k) = \alpha + \langle h, k \rangle = \rho_h^U(\alpha 1 + k):$$

Proof

Let ρ $S(V_1^*) \setminus \partial_e(V_1^*)$ such that ρ has a unique extension in W_1^*. Let u be the support tripotent of ρ in V. By Theorem 1.10.2, u cannot be minimal in V and so must be unitary in both V and W. Let E and F, respectively, denote the real spin factors $V_2(u)_{sa}$ and $W_2(u)_{sa}$. Then, by restriction, ρ $S(E) \setminus P(E)$ with unique extension in $S(F)$. Hence, by Lemma 3.5.3, $E = F$ and thus $V = W$. □

Proposition 3.5.5

Let M be a JBW-subtriple of a JBW*-triple N. Let $\rho = \square\, \lambda_n \rho_n$ be a σ*

n

Thus,

$$\|\psi\| = \sum_{k+1}^{\infty} \lambda_n \quad \text{and similarly} \quad \|\phi\| = \sum_{1}^{k} \lambda_n.$$

It follows that

$$\bar{\phi} = \sum_{1}^{k} \lambda_n \tau_n \quad \text{and} \quad \bar{\psi} = \sum_{k+1}^{\infty} \lambda_n \tau_n$$

are norm preserving extensions of ϕ and ψ, respectively, in N_*.

Now let ϕ' be any extension of ϕ in N_* such that $\|\phi'\| = \|\phi\|$. Then $\phi' + \bar{\psi}$ extends ρ and has norm one, since

$$1 = \|\rho\| \leq \|\phi' + \bar{\psi}\| = \|\phi\| + \|\psi\| = 1.$$

Hence, by uniqueness, we have

$$\phi' + \bar{\psi} = \bar{\rho} = \bar{\phi} + \bar{\psi}.$$

Therefore, $\phi' = \bar{\phi}$, which proves that $\bar{\phi}$ is the unique norm preserving extension of ϕ in N_*.

Finally, it follows from the above that, for each n, $\lambda_n \tau_n$ is the unique norm preserving extension of $\lambda_n \rho_n$. Hence, τ_n is the unique norm one extension of each ρ_n. This completes the proof. □

We can now prove the key result of this section. In the statement below we note that rank$(C) > 1$.

Theorem 3.5.6

Let C be a Cartan factor and JBW-subtriple of a JBW*-triple N. Suppose there exists $\rho \in S(C_{*,1}) \setminus \partial_e(C_{*,1})$ with unique extension in $N_{*,1}$. Then C is an inner ideal of N.*

Proof

Let $\rho \in S(C_{*,1}) \setminus \partial_e(C_{*,1})$ such that ρ has unique norm one extension $\bar{\rho}$ in $N_{*,1}$. Since every Cartan factor is atomic (Theorem 1.11.4(a)(i)), by Theorem 1.12.1(b) ρ is a σ-convex sum

$$\rho = \sum \lambda_n \rho_n, \quad \text{where } \lambda_n > 0 \text{ for each } n \text{ and } \sum \lambda_n = 1 \qquad (\ )$$

of at least two mutually orthogonal elements $\rho_n \in \partial_e(C_{*,1})$.

Consider now

$$\tau = \lambda_1\rho_1 + \lambda_2\rho_2 \quad \text{and} \quad \sigma = \tau / \|\tau\|.$$

By Proposition 3.5.5, ρ_1 has a unique extension in $N_{*,1}$. Therefore, by Lemma 3.3.1 and Lemma 3.3.2, there is a Cartan factor and weak* closed ideal D of N such that D contains C. It follows that ρ_1 has unique extension in $D_{*,1}$. Hence, the inclusion

$$C \subseteq D$$

satisfies the equivalent conditions of Theorem 3.3.4. In particular, with

$$u = s(\rho_1) + s(\rho_2), \quad \text{we have,} \quad C_2(u) \subseteq D_2(u)$$

is an inclusion of spin factors.

Further, by Proposition 3.5.5, σ has a unique norm one extension $\bar{\sigma}$ in D_*. Since $\sigma(u) = \bar{\sigma}(u) = 1$, we have

$$\sigma = \sigma \circ P_2(u) = \bar{\sigma} = \bar{\sigma} \circ P_2(u)$$

and we see that $\bar{\sigma}|_{D_2(u)}$ is the unique norm one extension of $\sigma|_{C_2(u)}$ in $D_2(u)_*$. Hence, by Lemma 3.5.4, $C_2(u) = D_2(u)$.

Now, applying Corollary 3.3.10, we see that either $C = D$ or C and D are hermitian, rectangular or symplectic and of the same generic type. In the latter event, C is an inner ideal of D by Theorem 3.4.8. Hence, C is an inner ideal of N, as required. □

In the context of Cartan factors, the results of sections 3.3 and 3.4 combined with Theorem 3.5.6 settle a number of unique extension questions and provide new characterisations of weak* closed inner ideals.

Theorem 3.5.7

Let C be a Cartan factor and JBW-subtriple of a JBW*-triple N. Suppose that* $\mathrm{rank}(C) > 1$. *Then the following are equivalent.*

(a) *There exists* $\rho \quad S(C_{*,1}) \setminus \partial_e(C_{*,1})$ *with unique extension in* $S(N_{*,1})$.

(b) *Every* $\rho \quad S(C_{*,1})$ *has unique extension in* $S(N_{*,1})$.

(c) *Every* $\rho \quad S(C_1^*)$ *has unique extension in* $S(N_1^*)$.

(d) *C is an inner ideal of N.*

Proof

The implication (a)   (b) follows from Theorem 3.5.6. The conditions (b), (c) and (d) are equivalent by Theorem 1.8.3. □

3.6 Von Neumann Algebras

We shall proceed to interpret our above unique extension results in the context of Cartan subfactors of von Neumann algebras. We recall the following proved in [32, Lemma 3.2].

Lemma 3.6.1

Let u be a partial isometry in a von Neumann algebra W. Then

$$W_2(u) = uu^*Wu^*u \quad (= uWu)$$

is a von Neumann algebra with respect to multiplication and involution given, respectively, by

$$a.b = au^*b \text{ and } a \quad = ua^*u.$$

Theorem 3.6.2

Let W be a von Neumann algebra and C a JBW-subtriple of W, where C is a Cartan factor. Suppose there exists $\rho \quad S(C_{*,1}) \setminus \partial_e(C_{*,1})$ with unique extension in $W_{*,1}$. Then we have the following.*

Theorem 3.6.3

Let $D = B(H)$, where H is a complex Hilbert space, and let C be linearly isometric to a von Neumann algebra. Then there exists $\rho \quad \partial_e(C_{*,1})$ with unique extension in $\partial_e(D_{*,1})$ if and only if $C = eB(H)f$ where e and f are von Neumann equivalent projections in D.

Proof

Since C and D have the same generic type (rectangular), C is an inner ideal of D by Theorem 3.4.8. In addition, by assumption, C contains a unitary tripotent u. Therefore,

$$C = C_2(u) = D_2(u).$$

But u is a partial isometry of $B(H)$ and we have

$$D_2(u) = uu^*B(H)u^*u. \qquad \square$$

Theorem 3.6.4

Let W be a von Neumann algebra and C be a JBW-subtriple of W*

Chapter 4

Unique Dual Ball Extensions

4.1 Introduction

Let C be a Cartan factor contained as a JBW*-subtriple in a JBW*-algebra M. In Chapter 3, it was shown that the existence of a single functional ρ in $\partial_e(C_{*,1})$ with unique extension in $M_{*,1}$ is sufficient to show that *every*

4.2 Local Theory

4.2.1 Let A be a JB*-triple and let $\rho \quad \partial_e(A_1^*)$. Since $s(\rho)$ is a minimal tripotent of A^{**}, the weak* closed ideal, A_ρ^{**}, of A^{**} generated by $s(\rho)$ is a Cartan factor [21]. Let

$$P_\rho : A^{**} \longrightarrow A_\rho^{**}$$

be the canonical weak* continuous M-projection from A^{**} onto A_ρ^{**}. We have

$$A^{**} = A_\rho^{**} \quad {}_\infty (\ker P_\rho).$$

Thus, when $(A_\rho^{**})_*$ is identified with its canonical isometric image in A^* we have

$$A^* = (A_\rho^{**})_* \quad {}_1 J_*$$

where $J = \ker P_\rho$, the orthogonal complementary weak* closed ideal of A_ρ^{**} in A^{**}. We shall often make such identification.

4.2.2 Let $\rho, \tau \quad \partial_e(A_1^*)$. Since A_ρ^{**}, A_τ^{**} are weak* closed ideals of A^{**} and are Cartan factors we have

(a) $A_\rho^{**} = A_\tau^{**}$ or $A_\rho^{**} \cap A_\tau^{**} = \{0\}$.

Thus we either have equality or A_ρ^{**} is orthogonal to A_τ^{**}. Further, for the same reasons we have

(b) $A_\rho^{**} = A_\tau^{**}$ if and only if $s(\rho) \quad A_\tau^{**}$.

(c) If $A_\rho^{**} = A_\tau^{**}$ we shall say that ρ and τ are equivalent and write $\rho \quad \tau$.

4.2.3 4.2.2(c) defines an equivalence relation on $\partial_e(A_1^*)$. The equivalence class of $\rho \quad \partial_e(A_1^*)$ shall be denoted by $[\rho]$. When, for $\rho \quad \partial_e(A_1^*)$, the predual of A_ρ^{**} is canonically identified as an ${}_1$-summand of A^* as indicated in 4.2.1, this gives

$$[\rho] = \partial_e((A_\rho^{**})_{*,1}).$$

4.2.4 Since $\{s(\rho) : \rho \quad \partial_e(A_1^*)\}$ is the set of minimal tripotents of A^{**}, 4.2.2(a) implies that

(a) A_{at}^{**} is the $_\infty$-sum of the distinct A_ρ^{**} as ρ ranges over $\partial_e(A_1^*)$.

Let I be a norm closed inner ideal of A. Since all minimal tripotents of I^{**} are minimal in A^{**} we have

(b) $I_{at}^{**} = A_{at}^{**} \cap I^{**}$.

Let $\rho \quad \partial_e(I_1^*)$ and identify ρ with its unique extension in $\partial_e(A_1^*)$. Since, by Theorem 1.11.4 (a)(iv), $A_\rho^{**} \cap I^{**}$ is a Cartan factor and contains I_ρ^{**} as a weak* closed ideal we have

(c) $I_\rho^{**} = A_\rho^{**} \cap I^{**}$.

Lemma 4.2.5

Let A be a JB-subtriple of a JB*-triple B. Let $\rho \quad \partial_e(A_1^*)$ and suppose that ρ has unique extension $\bar{\rho} \quad \partial_e(B_1^*)$. Then $s(\rho) = s(\bar{\rho})$, $A_\rho^{**} \quad B_{\bar{\rho}}^{**}$ and every minimal tripotent of A_ρ^{**} is minimal in $B_{\bar{\rho}}^{**}$.*

Proof

We have that A_ρ^{**} is a Cartan subfactor of the JBW*-triple B^{**}

Let A be a JB*-subtriple of a JB*-triple B. Lemma 4.2.5 shows that whenever there exists $\rho \quad \partial_e(A_1^*)$ with unique extension $\overline{\rho}$ in $\partial_e(B_1^*)$, then every element of $\partial_e(A_1^*)$ equivalent to ρ has unique extension in $\partial_e(B_1^*)$. We record a list of equivalent conditions in the statement below, which is companion to Theorem 3.3.4.

Theorem 4.2.6

Let $\rho \quad \partial_e(A_1^*)$, where A is a JB-subtriple of a JB*-triple B. Let $\overline{\rho}$ be an extension of ρ in B_1^*. Then the following are equivalent.*

(a) *$\overline{\rho}$ is the unique extension of ρ in B_1^*.*

(b) *$\overline{\rho}$ is the unique extension of ρ in $\partial_e(B_1^*)$.*

(c) *$s(\rho) = s(\overline{\rho})$.*

(d) *$A_\rho^{**} \quad B_{\overline{\rho}}^{**}$ and every minimal tripotent of A_ρ^{**} is minimal in $B_{\overline{\rho}}^{**}$.*

(e) *$K(A_\rho^{**})$ has the extreme extension property in $K(B_{\overline{\rho}}^{**})$.*

(f) *Every element of $[\rho]$ has unique extension in $\partial_e(B_1^*)$.*

Proof

(a) ⇔ (b) The equivalence of these conditions is is given by Corollary 2.4.2.

(b)   (c), (c)   (d). The required arguments are given by Lemma 4.2.5.

(d)   (e) This follows as in the proof of Theorem 3.3.4 (f)   (g).

(e)   (f) Assume that (e) holds. In particular, $A_\rho^{**} \quad B_{\overline{\rho}}^{**}$ by taking weak* closures and using Theorem 1.11.4 (a)(iii). Thus, (e) implies that every minimal tripotent of A_ρ^{**} is minimal in $B_{\overline{\rho}}^{**}$. Now let $\tau \quad [\rho]$ and let $\overline{\tau} \quad \partial_e(B_1^*)$ extend τ. Then $s(\tau)$ is minimal in $B_{\overline{\rho}}^{**}$ and $\overline{\tau}(s(\tau)) = 1$ so that, by Proposition 1.10.1, $\overline{\tau}$ is the unique element of $\partial_e(B_1^*)$ supporting $s(\tau)$.

(f)   (a) This is clear. □

If A, B, ρ and $\bar{\rho}$ are as in Theorem 4.2.6 and satisfy any of its equivalent conditions, then the results of Chapter 3 can be appealed to in order to study the relative structures of the arising Cartan factors A_{ρ}^{**} and $B_{\bar{\rho}}^{**}$. In particular, let us note that in these circumstances, the inclusion

$$A_{\rho}^{**} \qquad B_{\bar{\rho}}^{**}$$

satisfies the equivalent conditions of Theorem 3.3.4, a fact employed several times in the proof of Theorem 4.2.7, below.

Since there are many cases to consider giving rise to a formal statement of inordinate length, we shall prove each case in the list prior to the statement and proof of the one that follows.

Theorem 4.2.7

Let A be a JB-subtriple of a JB*-triple B. Let $\rho \quad \partial_e(A_1^*)$ such that ρ has unique extension $\bar{\rho}$ in $\partial_e(B_1^*)$.*

Case I A_{ρ}^{**} or $B_{\bar{\rho}}^{**}$ exceptional.

(a) *If $A_{\rho}^{**} = B_{1,2}$, then $B_{\bar{\rho}}^{**} = B_{1,2}$ or M_3^8.*

(b) *If $A_{\rho}^{**} = M_3^8$, then $B_{\bar{\rho}}^{**} = M_3^8$.*

(c) *If $B_{\bar{\rho}}^{**} = B_{1,2}$, then A_{ρ}^{**} is a rank 2 subfactor of $B_{\bar{\rho}}^{**}$ or a Hilbert space inner ideal of $B_{\bar{\rho}}^{**}$.*

(d) *If $B_{\bar{\rho}}^{**} = M_3^8$, then A_{ρ}^{**} is a rank 2 or 3 subfactor of $B_{\bar{\rho}}^{**}$ or a Hilbert space inner ideal of $B_{\bar{\rho}}^{**}$.*

Proof

(c) Suppose that $B_{\bar{\rho}}^{**} = B_{1,2}$. By Proposition 3.3.7, any rank 2 subfactor of $B_{\bar{\rho}}^{**}$ will have the extreme extension property in $B_{\bar{\rho}}^{**}$. Hence A_{ρ}^{**} must either be of this form, or A_{ρ}^{**} has rank 1 and hence is a Hilbert space. In the latter case, A_{ρ}^{**} is an inner ideal of $B_{\bar{\rho}}^{**}$, by Corollary 3.4.3.

(d) This is almost identical to the previous case. If $B_{\bar{\rho}}^{**} = M_3^8$, then there is the added possibility that A_{ρ}^{**} can be a rank 3 subfactor of $B_{\bar{\rho}}^{**}$. The rest of the proof is identical to that of (c). □

Case II A_{ρ}^{**} and $B_{\bar{\rho}}^{**}$ non-exceptional, A_{ρ}^{**} is a Hilbert space.

In this case, A_{ρ}^{**} must be an inner ideal of $B_{\bar{\rho}}^{**}$. If $\dim(A_{\rho}^{**}) \square 2$, *then $B_{\bar{\rho}}^{**}$ cannot be hermitian (but can be a spin factor or a rectangular or symplectic factor).*

Proof

A_{ρ}^{**} has the extreme extension property in $B_{\bar{\rho}}^{**}$ and hence A_{ρ}^{**} is an inner ideal of $B_{\bar{\rho}}^{**}$, by Corollary 3.4.3. If $\dim(A_{\rho}^{**}) \square 2$ then A_{ρ}^{**} cannot be hermitian, by Lemma 3.4.4. In turn, $B_{\bar{\rho}}^{**}$ cannot be hermitian by Theorem 3.3.11(c). □

Case III A_{ρ}^{**} and $B_{\bar{\rho}}^{**}$ non-exceptional, A_{ρ}^{**} a spin factor.

(a) *If $A_{\rho}^{**} = V_n$, where $n \square 6$, then $B_{\bar{\rho}}^{**} = V_m$, $m \square n$.*

(b) *If $A_{\rho}^{**} = V_n$, where $n = 4$ or 5, then $B_{\bar{\rho}}^{**}$ is symplectic or $B_{\bar{\rho}}^{**} = V_m$ where $m \square n$.*

(c) *If $A_{\rho}^{**} = V_3$, then $B_{\bar{\rho}}^{**}$ is rectangular or symplectic or $B_{\bar{\rho}}^{**} = V_m$, $m \square 3$.*

(d) *If $A_{\rho}^{**} = V_2$, then $B_{\bar{\rho}}^{**}$ is hermitian, rectangular or symplectic or $B_{\bar{\rho}}^{**} = V_m$, $m \square 2$.*

Proof

Since A_ρ^{**} has the extreme extension propertchHinnceA

Proof

(a) If $B_{\bar{\rho}}^{**}$

(ii) $A_{\rho}^{**} = B_{\bar{\rho}}^{**} = V$, *where* V *is a spin factor.*

4.3 The Extreme Extension Property and Structure Spaces

In this section we examine the extreme extension property of a JB*-subtriple A in a JB*-triple B in connection with certain ‘structure spaces’ of primitive ideals and equivalence classes of dual ball extreme points associated with A and B.

We begin by formally stating in Theorem 4.3.1 a list of conditions equivalent to the JB*-triple extreme extension property. The proof can be seen by inspection of the *individual* extreme extension property given in Theorem 4.2.6 and only brief indications are required. However, we shall continue to raid Theorem 4.2.6 for information when studying the full extreme extension property.

Theorem 4.3.1

Let A be a JB-subtriple of a JB*-triple B. Then the following are equivalent.*

(a) *A has the extreme extension property in B.*

(b) *Each $\rho \quad \partial_e(A^*_1)$ has a unique extension in B^*_1.*

(c) *If $\rho \quad \partial_e(A^*_1)$, then $s(\rho)$ is a minimal tripotent of B^{**}.*

4.3.2 Let A be a JB*-triple. A norm closed ideal I of A is said to be *primitive* if it is the largest norm closed ideal contained in $\ker\rho$, for some $\rho \in \partial_e(A_1^*)$. We write $\mathrm{Prim}(A)$ for the set of primitive ideals of A with the *structure topology* (or hull-kernel topology), derived as follows. Given $E \subset A$ and $S \subset \mathrm{Prim}(A)$, the *hull* of E is

$$h(E) = \{I \in \mathrm{Prim}(A) : E \subset I\}$$

and the *kernel* of S is

$$k(S) = \bigcap_{I \in S} I.$$

The hulls form the closed sets of the *structure topology* on $\mathrm{Prim}(A)$ and the closure of a subset $S \subset \mathrm{Prim}(A)$ is $hk(S)$ [3, Proposition 3.2]. The *structure space* of a JB*-triple A is $\mathrm{Prim}(A)$ endowed with the structure topology.

The map $I \to h(I)$ defines a bijection between the norm closed ideals of a JB*-triple A and the closed sets of $\mathrm{Prim}(A)$. For each norm closed ideal I of A we have the homeomorphisms

$$\begin{array}{rcl} h(I) & \longrightarrow & \mathrm{Prim}(A) \\ P & \longrightarrow & P/I \end{array}$$

and

$$\begin{array}{rcl} \mathrm{Prim}(A) \setminus h(I) & \longrightarrow & \mathrm{Prim}(I) \\ P & \longrightarrow & P \cap I \end{array}.$$

In fact, the second map is a homeomorphism even when I is a norm closed inner ideal of A [14, 3.3].

4.3.3 A *Cartan factor representation* of a JB*-triple A is a triple homomorphism $\pi : A \to C$, where C is a Cartan factor and $\overline{\pi(A)}^{w} = C$. We write $\mathcal{C}(A)$ for the set of Cartan factor representations of A.

Let A be a JB*-triple. Recall that the weak*-closed ideal A_ρ^{**} in A^{**} generated by $s(\rho)$ is a Cartan factor and we have the canonical weak* continuous M-projection

$$P_\rho : A^{**} \longrightarrow A_\rho^{**}$$

of A^{**} onto A_ρ^{**} (4.2.1). The restriction

$$\pi_\rho : A \longrightarrow A_\rho^{**}$$

of P_ρ is a Cartan factor representation of A. Essentially, all Cartan factor representations arise in this way (see Theorem 4.3.4(b) below). We note that P_ρ is the unique weak* continuous extension, to A^{**}, of π_ρ.

The next theorem is a combination of [14, 3.2] and [6, 3.6].

Theorem 4.3.4

Let A be a JB-triple.*

(a) *If $\rho \quad \partial_e(A_1^*)$ then $\ker \pi_\rho$ is the largest norm closed ideal of A in $\ker \rho$.*

(b) *Let $\pi : A \to C$ be a Cartan factor representation of A. Then there exists $\rho \quad \partial_e(A_1^*)$ and a surjective isometry $\phi : A_\rho^{**} \to C$ such that $\phi \circ \pi_\rho = \pi$.*

(c) $\mathrm{Prim}(A) = \{\ker \pi_\rho : \rho \quad \partial_e(A_1^*)\} = \{\ker \pi : \pi \quad \mathcal{C}(A)\}$.

4.3.5 Let A be a JB*-triple and let $\rho, \tau \quad \partial_e(A$

4.3.6 The function

$$\begin{aligned} \theta_A : \hat{A} &\longrightarrow \mathrm{Prim}(A) \\ [\rho] &\longrightarrow \ker \pi_\rho \end{aligned}$$

is well-defined. Indeed, if $\rho, \tau \quad \partial_e(A_1^*)$ with $[\rho] = [\tau]$, then $\pi_\rho = \pi_\tau$ (4.3.5) and it follows that $\ker \pi_\rho = \ker \pi_\tau$. Clearly θ_A is surjective. Define a topology on $\hat{A}$ to be $\{\theta_A^{-1}(U) : U \text{ is open in } \mathrm{Prim}(A)\}$. Then, by definition, θ_A is open and continuous.

Now define

$$\begin{aligned} \phi_A : \partial_e(A_1^*) &\longrightarrow \hat{A} \\ \rho &\longrightarrow [\rho] \end{aligned}$$

and put $\psi_A = \theta_A \circ \phi_A$. We shall need Proposition 4.3.7, which can be found in [17].

Proposition 4.3.7

If A is a JB-triple then $\psi_A : \partial_e(A_1^*) \to Prim(A)$ is open and continuous.*

Lemma 4.3.8

Let A be a JB-triple. Then $\phi_A : \partial_e(A_1^*) \to \hat{A}$ is open and continuous.*

Proof

Let U be an open subset of $\mathrm{Prim}(A)$. Then

$$\phi_A^{-1}(\theta_A^{-1}(U)) = (\theta_A \circ \phi_A)^{-1}(U) = \psi_A^{-1}(U)$$

is open in $\partial_e(A_1^*)$, proving continuity.

Now let U be an open subset of $\partial_e(A_1^*)$. Then

$$\theta_A(\phi_A(U)) = \psi_A(U)$$

is open in $\mathrm{Prim}(A)$. Hence,

$$\phi_A(U) = \theta_A^{-1}(\psi_A(U))$$

is open in $\hat{A}$ and it follows that ϕ_A is open. □

Lemma 4.3.9

Let A be a JB-subtriple of a JB*-triple B and suppose that A has the extreme extension property in B. Then the functions*

$$\begin{aligned} \beta : \hat{A} &\longrightarrow \hat{B} \\ [\rho] &\longrightarrow [\overline{\rho}] \end{aligned}$$

and

$$\begin{aligned} \gamma : Prim(A) &\longrightarrow Prim(B) \\ \ker \pi_\rho &\longrightarrow \ker \pi_{\overline{\rho}} \end{aligned}$$

are well-defined.

Proof

Let $\rho, \tau \quad \partial_e(A_1^*)$ such that $\rho \quad \tau$. Since A has the extreme extension property in B,

$$s(\overline{\rho}) = s(\rho) \quad A_\rho^{**} = A_\tau^{**} \quad B_{\overline{\tau}}^{**}$$

by 4.3.5 and Lemma 4.2.5 and proof. Hence, $\overline{\rho} \quad \overline{\tau}$ and β is well-defined.

For γ, let $\rho, \tau \quad \partial_e(A_1^*)$ such that $\ker \pi_\rho = \ker \pi_\tau$. It follows that $\rho \quad \tau$ and, as in the previous paragraph, we have $\overline{\rho} \quad \overline{\tau}$. Thus $\ker \pi_{\overline{\rho}} = \ker \pi_{\overline{\tau}}$, which completes the proof. □

Lemma 4.3.10

Let U, V, X, Y be topological spaces. If the diagram below commutes, where f and g are open and continuous, and h is continuous, then k is continuous.

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "h"'] & V \arrow[d, "k"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

Proof

Let $A \subseteq Y$ be open. By continuity of h and g, $h^{-1}(g^{-1}(A))$ is open. Thus, since the diagram commutes,

$$f^{-1}(k^{-1}(A)) \;=\; (k \circ f)^{-1}(A) \;=\; (g \circ h)^{-1}(A) \;=\; h^{-1}(g^{-1}(A))$$

is open. Finally, since f is open,

$$k^{-1}(A) \;=\; f(f^{-1}(k^{-1}(A)))$$

is open. $\square$

Theorem 4.3.11

Let A be a JB-subtriple of a JB*-triple B and suppose that A has the extreme extension property in B. Then the following diagram commutes.*

$$\begin{tikzcd}
\partial_e(A_1^*) \arrow[r, "\phi_A"] \arrow[d, "\alpha"'] & \hat{A} \arrow[r, "\theta_A"] \arrow[d, "\beta"] & Prim(A) \arrow[d, "\gamma"] \\
\partial_e(B_1^*) \arrow[r, "\phi_B"] & \hat{B} \arrow[r, "\theta_B"] & Prim(B)
\end{tikzcd}$$

All of the maps are continuous and the horizontal maps are open.

Proof

Subsection 4.3.6 and Lemma 4.3.8 show that the horizontal maps are open and continuous. By Proposition 2.4.7, α is continuous. Finally, two applications of Lemma 4.3.10 gives that β and γ are continuous. $\square$

We remark that Theorem 4.3.11 extends [13, Proposition 4.1].

4.4 Unique Extension of Cartan Factor and Atomic Functionals

4.4.1 Cartan Factor Functionals

Definition Let τ $S(A_1^*)$, where A is a JB*-triple. Then τ is defined to be a *Cartan factor functional* of A if the weak* closed ideal of A^{**} generated by $s(\tau)$ is a Cartan factor.

The set of all Cartan factor functionals of A is denoted by $C_e(A_1^*)$. We note that

$$\partial_e(A$$

then via the usual identification

$$C_{i,*} \;=\; K(C_i)^* \;=\; \{\rho \quad A^* : s(\rho) \quad C_i\}$$

we have

$$C_e(A_1^*) \;=\; \square \; S(C_{i,*,1}).$$

Since a Cartan factor C is a Hilbert space if and only if all elements of $S(C_{*,1})$ are extreme points, the next proposition is clear from the above remarks and the Cartan factor representation theory stated in Theorem 4.3.4 (b).

Proposition 4.4.3

Let A be a JB-triple. Then $C_e(A_1^*) = \partial_e(A_1^*)$ if and only if all Cartan factor representations of A are onto Hilbert spaces.*

Proposition 4.4.4

Let A be a JB-subtriple of a JB*-triple B. Let $\tau \quad C_e(A_1^*)$ such that $s(\tau) \quad A_\rho^{**}$, where $\rho \quad \partial_e(A_1^*)$. Suppose that τ has unique extension $\overline{\tau}$ in B_1^*. We have the following.*

(a) *ρ has unique extension $\overline{\rho}$ in B_1^*.*

(b) *If $\tau \;/\; \partial_e(A_1^*)$, then A_ρ^{**} is an inner ideal of $B_{\overline{\rho}}^{**}$.*

(c) *$s(\tau) = s(\overline{\tau})$.*

(d) *$\overline{\tau} \quad C_e(B_1^*)$.*

Proof

(i) If $\tau \quad \partial_e(A_1^*)$, then $\tau \quad [\rho]$ and (a), (c) and (d) (since $\overline{\tau} \quad \partial_e(B_1^*)$) are immediate from the equivalent conditions of Theorem 4.2.6.

(ii) Suppose now that τ / $\partial_e($

Let $\rho \quad \partial_e(A_1^*)$ and let $\overline{\rho}$ be its unique extension in $\partial_e(B_1^*)$ so that, by application of Theorem 4.2.6 we have

$$C \quad D,$$

where $C = A_\rho^{**}$ and $D = B_{\overline{\rho}}^{**}$.

Let $\phi \quad S(C_{*,1})$ and let ψ be an extension of ϕ in $S(D_{*,1})$. Then regarding $S(C_{*,1})$ and $S(D_{*,1})$ as being contained in $\mathcal{C}_e(A_1^*)$ and $\mathcal{C}_e(B_1^*)$, respectively, as noted above via the usual identifications, we see that ψ must be the unique extension of ϕ in $\mathcal{C}_e(B_1^*)$. Consequently, ψ is the unique norm one extension of ϕ in D_*. Hence, by Theorem 1.8.3, C is an inner ideal of D and therefore is an inner ideal of B_{at}^{**}, as required.

(c) (a) Suppose that (c) holds. We have

$$A_{at}^{**} = \quad J_i \quad {}_{\infty},$$

where each J_i is a weak* closed inner ideal of B_{at}^{**}. Each J_i is a weak* closed ideal of A_{at}^{**}. Let $\tau \quad \mathcal{C}_e(A_1^*)$. Then $s(\tau) \quad C$, and so we may suppose $\tau \quad S(C_{*,1})$, for some Cartan factor $C = A_\rho^{**}$ with $\rho \quad \partial_e(A_1^*)$. Now there exists i such that

$$C \cap J_i = \{0\} \quad \text{and so} \quad C \quad J_i.$$

Therefore, C is a weak* closed ideal of B^{**}. Hence, τ has unique extension in B_1^*.

(c) $\Leftrightarrow$ (d) This is clear from the above. $\square$

Numerous examples have been seen of a JB*-subtriple A having the extension property in a JB*-triple B when condition (c) of Theorem 4.4.5 fails and so that A fails to have the Cartan extension property in B.

In each of the following sample of corollaries, the proofs of which can be read from Corollary 4.2.8 and Theorem 4.4.5(c),

A is a JB-subtriple of a JB*-triple B.*

The first can also be deduced using Proposition 4.4.3

Corollary 4.4.6

Suppose that all Cartan factor representations of A are onto Hilbert spaces. Then A has the extreme extension property in B if and only if A has the Cartan extension property in B.

Corollary 4.4.7

Let A have the extreme extension property in B, let all Cartan factor representations of A be onto factors of rank ≥ 2 and let all Cartan factor representations of B be symplectic. Then A has the Cartan extension property in B if and only if all Cartan factor representations of A are symplectic.

Corollary 4.4.8

If A has the extreme extension property in B, then A has the Cartan extension property in B in each of the following cases.

(a) *All Cartan factor representations of A are symplectic.*

(b) *All Cartan factor representations of B are hermitian.*

4.5 The Atomic Extension Property

Recall that, as defined in Section 1.12, the set of atomic functionals of a JB*-triple A is the norm closed subspace of A^*

$$\{\rho \quad A^* : s(\rho) \quad A^{**}_{at}\}.$$

In particular, all Cartan factor functionals of A are atomic functionals.

We begin with a unique extension criterion for *individual* atomic functionals.

Theorem 4.5.1

Let A be a JB-subtriple of a JB*-triple B. Let ρ be an atomic functional in A^*. Then ρ has a unique norm preserving extension to an atomic functional in B^* if and only if ρ has a unique norm preserving extension in B^*.*

Proof

Some of the ingredients are already contained in the proof of Proposition 3.5.5. To be sure, let $\|\rho\| = 1$ and using Theorem 1.12.3 write ρ as a σ-convex sum

$$\rho = \sum \lambda_n \rho_n$$

of mutually orthogonal elements ρ_n in $\partial_e(A_1^*)$. For each n, let $\tau_n \in \partial_e(B_1^*)$ be an extension of ρ_n (using Lemma 2.4.1(b)) and put

$$\bar{\rho} = \sum \lambda_n \tau_n.$$

Then $\bar{\rho}$ is a norm one atomic functional in B^* extending ρ. In particular, we note this shows that

( ) every atomic functional in A^* has a norm preserving atomic extension in B^*.

Suppose now that ρ has unique norm one atomic extension in B^*. With

$$\phi = \sum_1^k \lambda_n \rho_n, \qquad \psi = \sum_{n>k} \lambda_n \rho_n$$

and

$$\bar{\phi} = \sum_1^k \lambda_n \tau_n, \qquad \bar{\psi} = \sum_{n>k} \lambda_n \tau_n$$

for each k, as in the proof of Proposition 3.5.5 we have

$$\|\phi\| = \|\bar{\phi}\| = \sum_1^k \lambda_n, \qquad \|\psi\| = \|\bar{\psi}\| = \sum_{n>k} \lambda_n$$

so that $\bar{\phi}$ and $\bar{\psi}$ are *atomic* extensions of ϕ and ψ, respectively, in B^*.

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Now, using ( ), let ϕ' be any norm preserving atomic ex4267-1(7ius)-1(ing)-426(()]TJF

4.5.2 Definition Let A be a JB*-subtriple of a JB*-triple B. Then A is said to have the *atomic extension property in* B if each atomic functional in A^* has a unique norm preserving extension to an atomic functional in B^*.

Equivalently, by Theorem 4.5.1, A has the atomic extension property in B if and only if every atomic functional in $S(A_1^*)$ has unique extension in B_1^*.

It is clear from the equivalence of (a), (b) and (c) in Theorem 4.3.1 that if A has the atomic extension property in B, then A has the extreme extension property in B and

$$A_{at}^{**} \qquad B_{at}^{**},$$

in which case, it follows that every element in the predual of A_{at}^{**} must have unique norm preserving extension in the predual of B_{at}^{**}

implies that A_{at}^{**} is an ideal of B_{at}^{**} and hence that A has the atomic extension property in B, by Proposition 4.5.3. $\square$

Corollary 4.5.5

Let A be a JB-subtriple of a JB*-triple B. If A has the atomic extension property in B, then A has the Cartan extension property in B.*

Proof

Proof

(a) ⇔ (b) This follows from Theorem 4.5.1.

(b) (c) Assume (b). We know that A has the Cartan extension property in B, by Corollary 4.5.5. Consider the extreme point unique extension map

$$\partial_e(A_1^*) \longrightarrow \partial_e(B_1^*) \quad (\rho \longrightarrow \overline{\rho})$$

and the corresponding map

$$\beta : \hat{A} \longrightarrow \hat{B} \quad ([\rho] \longrightarrow [\overline{\rho}])$$

of Lemma 4.3.9. Let $\rho, \tau \quad \partial_e(A_1^*)$ such that $[\overline{\rho}] = [\overline{\tau}]$.

$$A_\rho^{**}, A_\tau^{**} \qquad B_{\overline{\rho}}^{**} \;=\; B_{\overline{\tau}}^{**},$$

denotes the unique extension map. Moreover, since

$$\beta : \hat{A} \longrightarrow \hat{B} \quad ([\rho] \longrightarrow [\bar{\rho}])$$

is injective, we have that, for $\rho, \tau \in \partial_e(A_1^*)$,

$$A_\rho^{**} = A_\tau^{**} \quad \text{implies} \quad B_{\bar{\rho}}^{**} = B_{\bar{\tau}}^{**}.$$

Thus, writing $\partial_e(A_1^*)$ as a union of mutually disjoint classes,

$$\partial_e(A_1^*) = \bigcup [\rho_i],$$

we have

$$A_{at}^{**} = \Big(\bigoplus A_{\rho_i}^{**} \Big)_\infty$$

and putting

$$J = \Big(\bigoplus B_{\bar{\rho}_i}^{**} \Big)_\infty,$$

we have that J is an ℓ_∞-sum of mutually orthogonal weak* closed ideals of B_{at}^{**}.

Consider an element of A_{at}^{**}

$$x = \bigoplus x_i,$$

where each $x_i \in A_{\rho_i}^{**}$. Then, by the above,

$$\{xB^{**}x\} \subseteq \bigoplus \{x_i B_{\bar{\rho}_i}^{**} x_i\} \subseteq J.$$

Therefore, A_{at}^{**} is an inner ideal of J and hence of B_{at}^{**}. □

4.5.7 We remark that in Theorem 4.5.6 the condition (c) cannot be replaced with the condition

(c′) A has the extreme extension property in B and $\beta : \hat{A} \to \hat{B}$ is injective.

For example, with $2 \le n \le \infty$, if $A = S_n(\mathbb{C})$ and $B = M_n(\mathbb{C})$ the condition (c′) is clearly satisfied. But A is not an inner ideal of B.

4.6 JB*-Algebras and Unique Extension of Classes of States

4.6.1 The previous discussion applies to JB*-algebras (and C*-algebras) since these are examples of JB*-triples. The global order structure, as well as algebraic structure, of JB*-algebras leads to considerations that do not arise in general JB*-triples.

Let A be a JB*-algebra. Recall (1.10.3) the bijective correspondence

$$\rho \longrightarrow s(\rho)$$

between the set $P(A)$ of pure states of A and the minimal projections of A^{**}. We have

$$P(A) \;=\; S(A) \cap \partial_e(A_1^*) \qquad \partial_e(A_1^*)$$

and, as ρ ranges over $P(A)$ the weak* closed Cartan factor ideals of A^{**} are the type I factors $A^{**} \circ c(\rho)$, where $c(\rho)$ is the central support projection of ρ in A^{**}. In particular,

$$A_\rho^{**} \;=\; A^{**} \circ c(\rho), \qquad \text{for each } \rho \quad P(A).$$

If $\tau \quad \partial_e(A_1^*)$, then there exists $\rho \quad P(A)$ such that

$$A_\tau^{**} \;=\; A^{**} \circ c(\rho) \;=\; A_\rho^{**}$$

so that (in the notation of 4.3.5) $\tau \quad \rho$. As ρ ranges over $P(A)$ the sum of is a type I factor. The set of all type I factor states of A coincides with the set

$$S(A) \cap C_e(A_1^*).$$

The norm one atomic functionals of A that are states of A, the *atomic states* of A, comprises the subset of $S(A)$,

$$\{\rho \quad S(A) : \rho(z_A) = 1\}.$$

4.6.2 Let A be a JB*-algebra. A *type I factor representation* of A is a *-Jordan homomorphism

$$\pi : A \to M,$$

where M is a type I JBW*-algebra factor and $\pi(A)$ is weak* dense in M. In particular, every type I factor representation of A is a Cartan factor representation of A. Given $\rho \quad P(A)$, in the notation introduced in 4.3.3, the M-projection

$$P_\rho : A^{**} \longrightarrow A^{**} \circ c(\rho) \; (= A_\rho^{**})$$

and its restriction

$$\pi_\rho : A \longrightarrow A^{**} \circ c(\rho)$$

are given by the assignment

$$a \longrightarrow a \circ c(\rho).$$

Moreover, in direct correspondence with Theorem 4.3.4(b), if

$$\pi : A \longrightarrow M$$

is a type I factor representation of A, then there exists $\rho \quad P(A)$ and a surjective Jordan *-isomorphism

$$\phi : A_\rho^{**} \longrightarrow M$$

such that $\phi \circ \pi_\rho = \pi$.

4.6.3 Let A be a JB*-subalgebra of a JB*-algebra B. Then A is said to have

(a) the *pure extension property in* B if every pure state of A has unique extension to a pure state of B;

(b) the *type I factor extension property in* B if every type I factor state of A has unique extension to a type I factor state of B;

(c) the *atomic state extension property in* B if every atomic state of A has unique extension to an atomic state of B.

Lemma 4.6.4

Let A be a JB-subalgebra of a JB*-algebra B and let ϕ $S(A)$. Then every extension of ϕ in B_1^* is a state of B.*

Proof

Let ψ B_1^* such that ψ extends ϕ

Proof

Let τ be an extension of ρ in $\partial_e(B_1^*)$. Using Lemma 4.6.4, we have

$$\tau \quad S(B) \cap \partial_e(B_1^*) = P(B).$$

Hence, $\tau = \bar{\rho}$, by assumption. Thus, $\bar{\rho}$ is the unique ex(,)]Tm26 details are that, in this way, (b) ⇔ (d) by Theorem 4.2.6(a) ⇔ (c), and that (a) (b) since (as stated in the proof of Lemma 4.6.5) ρ has unique extension in $P(B)$ implies ρ has unique extension in $\partial_e(B_1^*)$ and hence in B_1^*, by Theorem 4.2.6(b) (a). □

Lemma 4.6.7

Let A be a JB-subalgebra of a JB*-algebra B and let A have the pure extension property in B. Then A has the extreme extension property in B*

Proof

Let u be a minimal tripotent of A^{**}. The weak* closed ideal of A^{**} generated by u is a type I JBW*-algebra factor of the form $A^{**} \circ c(\rho)$, for some ρ $P(A)$. Let $\overline{\rho}$ be the unique extension of ρ in $P(B)$. By Lemma 4.6.5, u is a minimal tripotent of $B^{**} \circ c(\overline{\rho})$ and hence is a minimal tripotent of B^{**}. Therefore, A has the extreme extension property in B by Theorem 4.3.1(e) (a). □

The following theorem is now obtained by combining Theorem 4.3.1 with Proposition 4.6.6 and Lemma 4.6.7.

Theorem 4.6.8

The following conditions are equivalent, for a JB-subalgebra A of a JB*-algebra B.*

(a) *A has the extreme extension property in B.*

(b) *A has the pure extension property in B.*

(c) *Each ρ $P(A)$ has unique extension in $S(B)$.*

(d) *If ρ $P(A)$, then $s(\rho)$ is a minimal projection of B^{**}.*

(e) *Each ρ $P(A)$ has an extension $\overline{\rho}$ $P(B)$ such that $s(\rho) = s(\overline{\rho})$.*

(f) *The minimal projections of A^{**} are minimal in B^{**}.*

Proof

The equivalence of (a), (c) and (d) is a direct consequence of Proposition 4.6.6 (a) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d). The equivalence of (d) and (f) follows from applications of Subsection 1.10.3.

Suppose A has the extreme extension property in B and let $\rho \quad P(A)$. By Lemma 4.6.4, ρ has unique extension in B_1^* and hence in $P(B)$, using Proposition 4.6.6. Thus, A has the pure extension property in B, proving (a) (b). The converse is Lemma 4.6.7.

Now assume that condition (a) holds and let $\rho \quad P(A)$. In particular, $\rho \quad \partial_e(A_1^*)$ so that there exists an extension $\bar{\rho}$ in $\partial_e(B_1^*)$ such that $s(\rho) = s(\bar{\rho})$. But $\bar{\rho}$ is a state of B, by Lemma 4.6.4. Hence $\bar{\rho} \quad P(B)$, so that condition (e) holds.

Finally, we show that (e) (d). Let $\rho \quad P(A)$. Condition (e) implies that there exists an extension $\bar{\rho}$ in $P(B)$ such that $s(\rho) = s(\bar{\rho})$. But $s(\bar{\rho})$ is a minimal projection of B^{**} and hence condition (d) holds. □

Let A be a JB*-algebra. A JB*-subalgebra I of A is an *hereditary* JB*-subalgebra of A if and only if whenever $x \quad A$ and $y \quad I$ such that

$$0 \,\square\, x \,\square\, y,$$

then $x \quad I$. Th6

of I^{**}. The following 'ordered' version of Theorem 1.8.3 is contained in [33]. It was proved for C*-algebras in [56].

Lemma 4.6.9

The following are equivalent for a JB-subalgebra I of a JB*-algebra A.*

(c) (d) Assume (c). Then A has the pure extension property in B (pure states are type I factor states and extend to pure states). Let $\rho \quad P(A)$ with unique extension $\overline{\rho} \quad P(B)$. Lemma 4.6.5 gives

$$A^{**} \circ c(\rho) \qquad B^{**} \circ c(\overline{\rho}) \qquad B^{**} \circ z_B.$$

Since all normal states of $A^{**} \circ c(\rho)$ and $B^{**} \circ c(\overline{\rho})$ correspond to type I factor states of A and B respectively (4.6.2), every normal state of $A^{**} \circ c(\rho)$ has unique extension to a normal state of $B^{**} \circ c(\overline{\rho})$. Hence, by Lemma 4.6.9, $A^{**} \circ c(\rho)$ is an hereditary subalgebra of $B^{**} \circ c(\overline{\rho})$ and hence of $B^{**} \circ z_B$.

(d) (e) This is clear.

(e) (a) This follows from Lemma 4.6.9 and Theorem 4.4.5. □

Theorem 4.6.11

Let A be a JB-subalgebra of a JB*-algebra B. Let ρ be an atomic state of A. Then ρ has unique extension to an atomic state of B if and only if ρ has unique extension in $S(B)$.*

Proof

This follows from Lemma 4.6.4 and Theorem 4.5.1. □

Theorem 4.6.12

The following are equivalent for a JB-subalgebra A of a JB*-algebra B.*

(a) *A has the atomic extension property in B.*

(b) *A has the atomic state extension property in B.*

(c) *Every atomic state of A has unique extension in $S(B)$.*

(d) *$A^{**} \circ z_A$ is an hereditary subalgebra of $B^{**} \circ z_B$.*

(e) *A has the type I factor extension property in B and $\beta : \hat{A} \to \hat{B}$ is injective.*

Chapter 5

Weak Sequential Convergence, Weakly Compact JB*-Triples and Unique Extensions

5.1 Introduction

The ostensible purpose of this chapter is to investigate and to resolve the extreme extension property of a separable JB*-triple A in a JBW*-triple M. As shall be shown, eventually, the JB*-triple in this case A turns out to be a c_0-sum of elementary JB*-triples. JB*-triples of this kind were studied under the name of *weakly compact* in [12], where several equivalent conditions were found. Along the way to establishing the weak compactness (in the above sense) of a separable JB*-triple having the extreme extension property in a JBW*-triple, and in order to bring into appropriate relief various structural implications, we prove a number of results of independent interest.

Weakly compact JB*-triples are aired in section 5.2. A new 'inner ideal' characterisation of weakly compact JB*-triples is given, appropriately in line with the unique extension theme of this thesis. The extreme extension property in the context of weakly compact JB*-triples is investigated and the links to the work of earlier chapters are exposed. The final section of the chapter is almost entirely devoted to a study of weak *sequential* convergence and weak* *sequential* convergence in the topological space $\partial_e(A_1^*)$ of a JB*-

triple A. It is shown that weak sequential convergence in $\partial_e(A_1^*)$ always implies norm convergence, first by proving the truth of this for all spin factors and then by exploiting work of [15] on the so-called Kadec-Klee property on the unit sphere of JB*-triple duals. Subsequent application of a deep result of [19] allows us to conclude that, for every JBW*-triple M, any sequence converging in $\partial_e(M_1^*)$ with respect to the weak* topology must converge in norm.

Concentrating upon separable JB*-triples thereafter, it is shown that in these cases the weak* topology is homeomorphic to the norm topology on $\partial_e(A_1^*)$ precisely when A is a weakly compact JB*-triple. This enables the taking of the final step in order to characterise the extreme extension property, and the atomic extension property, of separable JB*-triples in JBW*-triples.

5.2 JB*-Triples Generated by Minimal Tripotents

5.2.1 Given a JB*-triple A we shall denote by $K(A)$ the norm closed linear span of the minimal tripotents of A. If A has no minimal tripotents we write $K(A) = \{0\}$. This is consistent with the notation $K(C)$ for the elementary ideal of a Cartan factor C (but differs slightly from the usage in [12]). We remark that $K(A)$ is sometimes referred to as the *socle* of A. The ideal $K(A)$ and particularly the case when $K(A) = A$ (that is, when A is generated as a Banach space by its minimal tripotents) was investigated in [12]. See also [8] for a more general and algebraic study of related Jordan Banach *-algebras. We recall that for some family (possibly empty) (C_i) of Cartan factors

(a) $K(A) = (\bigoplus K(C_i))_0$ [12, Lemma 3.3]

and thus

(b) $K(A)^{**} = (\bigoplus K(C_i)^{**})_\infty = (\bigoplus C_i)_\infty$.

5.2.2 Let A be a JB*-triple. Then A is said to be a *weakly compact* JB*-triple [12] if for all $x \in A$, the conjugate linear operator

$$Q_x : A \longrightarrow A$$

is a weakly compact operator (that is, $Q_x(a_n)$ has a weakly convergent subsequence whenever (a_n) is a bounded sequence in A). An array of characterisations is given in [12, Theorem 3.4], including the fact that A is a weakly compact JB*-triple if and only if

$$D(x, x) : A \longrightarrow A$$

is weakly compact for all $x \in A$. For later use, other characterisations are listed in the following statement.

Theorem 5.2.3 [12, Theorem 3.4]
The following conditions are equivalent for a JB-triple A.*

(a) *A is weakly compact.*

(b) *A is an ideal of A^{**}.*

(c) *$K(A) = A$.*

(d) *A contains all minimal tripotents of A^{**} (that is, $K(A) = K(A^{**})$).*

Therefore a JB*-triple A is weakly compact if and only if

$$A = \Big(\bigoplus E_i\Big)_0,$$

is a c_0-sum of elementary ideals E_i of A. It is clear from the definition that every JB*-subtriple of a weakly compact JB*-triple is a weakly compact JB*-triple. On the other hand, it is of course possible for a weakly compact

JB*-triple to be a JB*-subtriple of a JB*-triple with no minimal tripotents. (For example, take a tripotent u in a continuous JBW*-triple B and put $A = \mathbb{C}u$.)

If A is a JB*-algebra, the condition (b) of Theorem 5.2.3 is equivalent to A_{sa} being an ideal of A

Proof

Let I be a norm closed inner ideal of A. Then I^{**} is a weak* closed inner ideal of A^{**}. Thus, by Theorem 1.8.6, there is a surjective weak* continuous structural projection

$$P : A^{**} \longrightarrow I^{**}.$$

By (c) and (d) of Theorem 5.2.3, $A = K(A^{**})$. Therefore

$$P(A) \quad A,$$

by Lemma 5.2.5. Therefore, P restricts to a structural projection on A and

$$P(A) \quad A \cap I^{**}$$

Proof

The equivalence of (a) and (b) is immediate from Corollary 2.4.9, and the implication (c) (b) was proved in Proposition 5.2.6. It remains to prove that (b) (c).

Suppose that the condition (b) holds. Let $x \quad A$. Let B denote the norm closed inner ideal $A(x)$ of A generated by x. By Theorem 1.7.1 we may regard B as a JB*-algebra with $x \quad B_+$. Let I be a an hereditary JB*-subalgebra of B. For some projection e of B^{**} we have

$$I^{**} = \{e \circ B^{**} \circ e\} = P_2(e)(B^{**}) = P_2(e)(A^{**}).$$

Since I is a norm closed inner ideal of A, by assumption there exists a surjective structural projection

$$P : A \longrightarrow I.$$

Since both

$$P^{**} : A^{**} \longrightarrow I^{**} \quad \text{and} \quad P_2(e) : A^{**} \longrightarrow I^{**}$$

are surjective structural projections (see 1.8.5), they are equal by uniqueness (Theorem 1.8.6). Hence, P and $P_2(e)$ agree on A. Therefore,

$$I = P(A) = P_2(e)(B) = \{e \circ B \circ e\}.$$

Therefore, $A(x)$ is weakly compact by Lemma 5.2.4 and so is contained in $K(A)$, since minimal tripotents of $A(x)$ are minimal in A. Hence, $K(A)$ contains all elements of A, which is therefore weakly compact by Theorem 5.2.3 (a) $\Leftrightarrow$ (c). $\square$

Concerning the extreme extension property involving weakly compact JB*-triples, the essential principles are already laid down in Chapters 3 and 4.

Proposition 5.2.8

Let A be a weakly compact JB-subtriple of a JB*-triple B. Then A has the Cartan extension property in B if and only if A is a c_0-sum of orthogonal norm closed inner ideals of B.*

Proof

Let A have the Cartan extension property in B. If A is an elementary JB*-triple then A^{**} is a Cartan factor, so that

$$C_e(A_1^*) = S(A_1^*).$$

In which case, by Theorem 4.4.5(b) (a), A has the extension property in B so that A is an inner ideal of B by Theorem 1.8.3. In general, A has the form (see 5.2.1)

$$A = \left(\bigoplus J_i \right)_0,$$

where the (J_i) are orthogonal norm closed elementary ideals of A. By the above each J_i is an inner ideal of B since each must have the Cartan extension property in B.

Conversely, suppose that

$$A = \left(\bigoplus J_i \right)_0,$$

where the J_i are orthogonal norm closed inner ideals of B. Then

$$A^{**} = \left(\bigoplus J_i^{**} \right)_\infty,$$

is an $_\infty$-sum of weak* closed inner ideals of B_{at}^{**}. Hence, A has the Cartan extension property in B, by Theorem 4.4.5(b) $\Leftrightarrow$ (c). □

Proposition 5.2.9

Let A be a weakly compact JB-subtriple of a JB*-triple B. Then the following are equivalent.*

(a) *A has the extreme extension property in B.*

(b) *Every minimal tripotent of A is a minimal tripotent of B.*

(c) *A has the extreme extension property in $K(B)$.*

Proof

(a) (b) By Theorem 4.3.1(a) (e), the condition (a) implies that the minimal tripotents of A^{**} are minimal in B^{**}. But all minimal tripotents of A^{**} are contained in A, by Theorem 5.2.3(d).

(b) (c) Since $A = K(A)$, by Theorem 5.2.3(c), this is immediate from the definition of $K(B)$.

(c) (a) This follows from the fact that $K(B)$ is a norm closed ideal of B (and so has the extension property in B). □

The next statement is an automatic consequence of the fact, as follows from 5.2.1(b), that A^{**} is an atomic JBW*-triple when A is a weakly compact JB*-triple.

Proposition 5.2.10

Let A be a weakly compact JB-subtriple of a JB*-triple B. Then A has the atomic extension property in B if and only if A is an inner ideal of B.*

5.2.11 Making a closer inspection of the situation where A is a weakly compact JB*-subtriple of a JB*-triple B such that A has the extreme extension property in B, by reducing to $K(B)$, it follows from Proposition 5.2.9 that it may as well be supposed that B is a weakly compact JB*-triple and thus, with (E_i) and (D_j) being the families of mutually orthogonal elementary ideals of A and B, respectively, we have

(a) $A = (\bigoplus E_i)_0$ and $B = (\bigoplus D_j)_0$.

We note that, for fixed j,

(i) *$A \cap D_j = \{0\}$ if and only if $E_i \cap D_j = \{0\}$ for all i;*

(ii) *if i is such that $E_i \cap D_j = \{0\}$, then $E_i \quad D_j$, and E_i has the extreme extension property in D_j.*

To see (i) note that if $A \cap D_j = \{0\}$ then it is a weakly compact ideal of A and therefore contains a minimal tripotent u of A. In that case, u must belong to some E_i, giving $E_i \cap D_j = \{0\}$. The converse is trivial.

To see (ii): $E_i \cap D_j$ is a norm closed ideal of the elementary JB*-triple E_i and thus, if it is non-zero, it must be all of E_i, giving $E_i \quad D_j$. In that case it is clear that every minimal tripotent of E_i is minimal in D_j.

Since the D_j for which $A \cap D_j = \{0\}$ make no contribution to the analysis, there is no harm in assuming that

(b) $A = (\bigoplus E_i)_0$, $B = (\bigoplus D_j)_0$, *where each $E_i \quad D_j$ for some (unique) j.*

(We remark that A can have many summands and B just one. See 5.2.13.)

The main structural results of Theorems 3.3.11, 3.4.8, 3.5.7 as well as Theorem 4.2.7 and Corollary 4.2.8, are available for a further analysis. We shall treat one example.

If F is a weakly compact JB*-triple, let F_{R} denote the c_0-sum of all hermitian elementary ideals of F, and let F_{de}

In Proposition 5.2.12 it is assumed that

> *A is a JB*-subtriple of a JB*-triple B, that A and B are weakly compact and that $A \cap D = \{0\}$, for every elementary ideal D of B.*

(b) Suppose that $B_{\mathrm{H}} = \{0\}$. We then have

$$A = A_{\mathrm{R}} \oplus A_{\mathrm{C}} \quad \text{and} \quad B = B_{\mathrm{R}} \oplus B_{\mathrm{C}}.$$

As in the proof of part (a), the same combination of Theorems 3.3.11(c) and 3.4.8 shows that every elementary ideal of A_{C} is an inner ideal of B_{C}.

(c) Since in this case

$$A = A_{\mathrm{R}} \quad \text{and} \quad B = B_{\mathrm{R}}$$

the result follows from Theorem 3.4.8 in a similar manner to that above. □

5.2.13 Examples

We note that in 5.2.11 (or 5.2.12) it is possible for A to have the unique extension property in B, where B is itself an elementary JB*-triple and A is not elementary. For instance, this is the case when

$$A = \left(\bigoplus \mathbb{C}u_i\right)_0 \quad \text{and} \quad B = K(C),$$

where C is a Cartan factor of rank ≥ 2 and (u_i) is a family of orthogonal minimal tripotents of cardinality ≥ 2 in C. For an example, where all factors are infinite dimensional and non-abelian consider a type I_∞ JW*-algebra factor M. Let $(e_i)_{i \in I}$ be an (infinite) orthogonal family of minimal projections in M such that

$$\sum_{I} e_i = 1.$$

Choose an infinite sequence (S_n) of infinite disjoint subsets of I such that

$$I = \bigcup_{n=1}^{\infty} S_n.$$

For each n, put

$$f_n = \sum_{S_n} e_i.$$

Then (f_n) is an orthogonal sequence of infinite rank projections in M, and $f_n \notin K(M)$ for all n. For each n, consider the type I_∞ factor

$$M_n = f_n M f_n.$$

Then

$$K(M_n) = f_n K(M) f_n$$

is a norm closed inner ideal of $K(M)$ for each n, and with

$$A = \Big(\bigoplus \sum K(M_n)\Big)_0 \quad \text{and} \quad B = K(M),$$

A has the extreme extension property in B.

5.3 Weak and Weak* Sequential Convergence and Unique Extensions

In this section we consider certain Banach space properties involving weak sequential convergence and weak* sequential convergence on the unit spheres of dual balls of JB*-triples and their restriction to sets of dual ball extreme points.

5.3.1 Definitions Let X be a Banach space. We continue to write

$$S(X) = \{x \in X : \|x\| = 1\}.$$

(a) X is said to have the *Kadec-Klee property* if weak sequential convergence in $S(X_1)$ implies norm convergence (that is, if (x_n) is a sequence in $S(X_1)$ and $x \in S(X_1)$ such that $x_n \to x$ in the weak topology then $\|x_n - x\| \to 0$);

(b) X^* is said to have the *weak* Kadec-Klee property* if weak* sequential convergence in $S(X_1)$ implies norm convergence (with the obvious meaning corresponding to that in parenthesis above).

In the context of JB*-triples and their dual spaces these properties have been studied in [1] and [15] from which the following is extracted for later operators in $B(H)$ and 1 is the identity element of $B(H)$, where H is an infinite dimensional Hilbert space. Hence, A^* has the Kadec-Klee property (since $A^{**} = B^{**} \quad {}_{\infty} C$) but A is not weakly compact.

In the sequel we study what amounts to weakening of the Kadec-Klee property and its weak* topological variation replacing $S(A_1^*)$ with the smaller set $\partial_e(A_1^*)$ for JB*-triples A. One benefit is the recovery of infinite dimensional spin factors excluded from the original theory (see Proposition 5.3.2(a), (b), (c)).

Lemma 5.3.3

Let J be a norm closed ideal in a JB-triple A. Let S be the subset of $\partial_e(A_1^*)$,*

$$S = \{\rho \quad \partial_e(A_1^*) : \rho(J) = 0\}.$$

Then S is weak closed in $\partial_e(A_1^*)$ if and only if*

$$A = J \quad I,$$

for some norm closed ideal I of A.

Proof

Suppose that S is weak* closed in $\partial_e(A_1^*)$. Then by Proposition 4.3.8 and notation we have that, for some norm closed ideal I of A,

$$\begin{aligned} \mathrm{Prim}(A) \setminus h(I) &= \psi_A(\partial_e(A_1^*) \setminus S) \\ &= \{\ker \pi_\rho : \rho \quad \partial_e(A_1^*) \cap J^\circ\} \\ &= h(J). \end{aligned}$$

Therefore,

$$\mathrm{Prim}(A) = h(I) \quad h(J) = h(I \cap J)$$

which gives $I \cap J = \{0\}$, so that I and J are orthogonal. In addition,

$$h(I + J) = h(I) \cap h(J) =$$

so that

$$A = I + J.$$

Conversely, if $A = I \quad J$, where I is a norm closed ideal of A, then

$$S = \{\rho \quad \partial_e(A_1^*) : \rho(J) = 0\} = \partial_e(A_1^*) \cap I^\circ$$

which is weak* closed in $\partial_e(A_1^*)$. □

Lemma 5.3.4

Let J be a norm closed ideal of a JB-triple A and let $\rho, \tau \quad \partial_e(A_1^*)$ such that $\rho(J) = \{0\}$ and $\tau(J) = \{0\}$. Then $\rho - \tau = 2$.*

Proof

This follows from 1.10.5 since, by Corollary 1.10.4, $s(\tau) \quad J^{**}$ and $s(\rho)$ lies in the orthogonal complement of J^{**} in A^{**}. □

Lemma 5.3.5

Let J be a weak closed ideal of a JBW*-triple M. Suppose that $\rho \quad \partial_e(M_{*,1})$ such that $s(\rho) \quad J$, and that (ρ_n)*

In Proposition 5.3.6 it is convenient to make use of the fact that [45] (see also [34, p60]) that a (complex) spin factor V can be constructed from a complex Hilbert space $(H, \ \cdot \ _2)$ and a conjugation $x \to \bar{x}$ on H by defining the triple product and norm given by

$$
\begin{aligned}
\{xyz\} &= \ x, y \ z + \ z, y \ x - \ x, \bar{z} \ \bar{y}, \\
x \ ^2 &= \ x
\end{aligned}
$$

Thus, since $\rho(x)u = P_2(u)(x)$, by Theorem 1.10.2,

$$\rho(x)u \;=\; 2\; x, u\; u \quad \text{and, similarly,} \quad \rho_n(x)u_n \;=\; 2\; x, u_n\; u_n \qquad (\dagger)$$

for each n. Therefore, for each n,

$$|(\rho_n - \rho)(x)| \;=\; 2|\; x, u_n - u\; | \;\square\; 2\; x\; {}_2\; u_n - u\; {}_2 \;\square\; 2\; x \quad u_n - u\; {}_2$$

and so

$$\rho_n -$$

If M is a spin factor then Proposition 5.3.6 implies that

$$\rho_n - \rho \quad \to 0.$$

If M is not a spin factor the same conclusion is deduced because by Lemma 5.3.2(a), M_* has the Kadec-Klee property. □

We shall now investigate weak* sequential convergence in $\partial_e(A_1^*)$ when A is a JB*-triple. Our aim is to show that when A is *separable* and the weak* topology coincides with the norm topology on $\partial_e(A_1^*)$ then A must be weakly compact. In order to achieve th397T

We now focus upon separable JB*-triples. The following notations are employed below. If τ is a topology on X^*, where X is a Banach space and $S \quad X^*$, (S, τ) denotes the topological subspace S of X^* with respect to τ. The weak, weak* and norm topologies on X^* are indicated by the symbols w, w^* and $\cdot$, respectively.

We shall need the following result of [47, Corollary], reproduced in [24, Theorem 10, p161].

Theorem 5.3.11

If X is a separable Banach space such that $\partial_e(X_1^*)$ is norm separable then X^* is norm separable.

Corollary 5.3.12

Let X be a separable Banach space such that whenever (ρ_n) is a sequence in $\partial_e(X_1^*)$ and $\rho \quad \partial_e(X_1^*)$ such that

$$\rho_n \to \rho \quad \text{in the weak* topology,}$$

we have

$$\rho$$

The structure of JB*-triples A for which A^* is norm separable was determined in [12]. In particular, by [12, Corollary 3.6] and the fact that if M is a closed subspace of a Banach space X such that X^* is norm separable, then $M^*(= X^*/M^\circ)$ is also norm separable, we have the following.

Lemma 5.3.13

Let A be a JB-triple such that A^* is norm separable. Then $K(B) = \{0\}$ for every non-zero JB*-subtriple B of A.*

Theorem 5.3.14

Let A be a separable JB-triple. Then the following are equivalent.*

(a) *If (ρ_n) is a sequence in $\partial_e(A_1^*)$ with weak* limit ρ $\partial_e(A_1^*)$, then*

$$\rho_n$$

contradicts Lemma 5.3.13. Hence, there is no such ideal and therefore S is not weak* closed in $\partial_e(A_1^*)$. Therefore there is a sequence (ρ_n) in S and $\rho \quad \partial_e(A_1^*) \cap K(A)^\circ$ such that

$$\rho_n \longrightarrow \rho \text{ in the weak* topology.}$$

But

$$\rho_n - \rho \quad = \ 2$$

for all n, by Lemma 5.3.4. This contradicts condition (a). Therefore, $K(A) = A$.

(b) ⇔ (c) The decomposition is countable because A is separable.

(b) (a) Let A be weakly compact. Then A is an ideal of A^{**}, by Theorem 5.2.3. In particular, A has the extension property in A^{**}. Consider the unique extension map

$$\partial_e(A_1^*) \longrightarrow \partial_e(A_1^{***}) \quad (\rho \longrightarrow \overline{\rho})$$

and recall that it is weak* continuous, by Proposition 2.4.7. Let

$$\rho_n \to \rho \text{ in } (\partial_e(A_1^*), w^*).$$

Then

$$\overline{\rho}_n \to \overline{\rho} \text{ in } (\partial_e(A_1^{***}), w^*) \qquad (\text{here } w^* \equiv \sigma(A^{***}, A^{**})).$$

Now applying Corollary 5.3.9 (with $M = A^{**}$) we have that

$$\overline{\rho}_n - \overline{\rho} \quad \to 0.$$

Hence,

$$\rho_n - \rho \quad \to 0. \qquad \square$$

5.4 Separable JB*-Subtriples of JBW*-Triples and Unique Extension Properties

In this final section we consider a separable JB*-subtriple A of a JBW*-triple M and characterise the various unique extension conditions (considered earlier in this thesis) of A in M in terms of the structure of A including that in relation to the structure of M.

Theorem 5.4.1

Let A be a separable JB-subtriple of a JBW*-triple M.*

(a) *A has the extreme extension property in M if and only if A is weakly compact and has the extreme extension property in $K(M)$.*

(b) *A has the Cartan factor extension property in M if and only if A is a c_0-sum of orthogonal norm closed inner ideals of $K(M)$.*

(c) *A has the atomic extension property in M if and only if A is an inner ideal of $K(M)$.*

Proof

(a) Suppose that A has the extreme extension property in M and let

$$\partial_e(A_1^*) \longrightarrow \partial_e(M_1^*) \quad (\rho \longrightarrow \overline{\rho})$$

be the unique extension map. By argument rehearsed in the proof of Theorem 5.3.14, if

$$\rho_n \to \rho \quad \text{in } (\partial_e(A_1^*), w^*)$$

then Proposition 2.4.7 implies that

$$\overline{\rho}_n \to \overline{\rho} \quad \text{in } (\partial_e(M_1^*), w^*)$$

so that by Corollary 5.3.9

$$\bar{\rho}_n - \bar{\rho} \quad \to 0$$

which implies that $\rho_n - \rho \quad \to 0$ and thus that A is weakly compact, by Theorem 5.3.14. Now Proposition 5.2.9 implies that A has the extreme extension property in $K(M)$.

Conversely, if A has the extreme extension property in $K(M)$ then it has the extreme extension property in M.

(b) Let A have the Cartan extension property in M. As previously observed (see the proof of Theorem 4.4.5 (b) (c)) A must have the extreme extension property in M. Thus A is weakly compact and

$$A \qquad K(M).$$

Since A therefore has the Cartan extension property in $K(M)$, A is of the claimed form by Proposition 5.2.8.

Conversely, if A has the stated decomposition, it must be weakly compact, because $K(M)$ is weakly compact, and therefore have the Cartan extension property in $K(M)$, by Proposition 5.2.8

(c) By similar argument, via (a), if A has the atomic extension property in M, then A has the atomic extension property in $K(M)$ and thus is an inner ideal of $K(M)$, using Proposition 5.2.10.

Conversely, any norm closed inner ideal of $K(M)$ is a norm closed inner ideal of M and thus must have the extension property in M and hence the atomic extension property in M, by Theorem 4.5.6. □

Finally, we shall record the 'JB*-algebra state' version of Theorem 5.4.1, the proof of which can now be read from Theorem 5.4.1 and the relevant results of Section 4.6, including Theorem 4.6.8, Theorem 4.6.10, Theorem 4.6.12 and Lemma 4.6.4 and Lemma 4.6.9.

Theorem 5.4.2

Let A be a separable JB-subalgebra of a JBW*-algebra M.*

(a) *A has the pure extension property in M if and only if A is weakly compact and has the pure extension property in $K(M)$.*

(b) *A has the type I factor extension property in M if and only if A is a c_0-sum of orthogonal hereditary JB*-subalgebras of $K(M)$.*

(c) *A has the atomic state extension property in M if and only if A is an hereditary JB*-subalgebra of $K(M)$.*

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