The Application of Optimal Control Theory

to Life Cycle Strategies.

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Dissertation submitted in partial fulfilment

for the Degree of Master of science

September 1993.

Abstract

The aim of this work, is to develop a numerical model, that could be used to in the calculation of optimal life cycle strategies of given organisms. The theory used from [6] assumes that the organisims have either a two phase 'bang-bang' life cycle strategy or a three phase life cycle strategy with the second phase being a singular arc.

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Introduction

The aim of this work is to develop a numerical method, which will find the optimal allocation of resources between growth and reproduction for a given organism and, hence, an organism's optimal life cycle strategy. A brief introduction to optimal control theory and life cycle strategies is given here. In addition an overview of the project's content and it's organisation is included.

1.1 What is Optimal ontrol Theory?

Optimal control theory arises from the consideration of physical systems, which are required to achieve a definite objective as "cheaply" as possible. The translation of the design objectives into a mathematical model gives rise to what is known as the control problem. The essential elements of a control problem are:-

- The system which is to be "controlled"
- A system objective
- A set of admissible "controls" (inputs)

- A performance functional which measures the effectiveness of a given "control action"

The system objective is a given state or set of states which may vary with time. Restrictions or constraints, as they are normally called, are placed on the set of controls (inputs) to the system; controls satisfying these constraints are said to belong to the set of admissible controls. A formal definition of the optimal control problem and associated theory of Pontryagin's principle is looked at in chapter two.

1.2 Life ycle Strategies

An organism's life cycle strategy is determined by the way in which it allocates energy between growth and reproduction. There are just two main ways in which organisms allocate energy. These are:-

- Determinate growth results in a bang-bang allocation strategy. All energy is initially allocated to growth until maturity, when maturity is reached all energy is switched to reproduction.

- Indeterminate growth results in a split energy allocation. This allows both growth and reproduction to take place simultaneously.

The general analytical model of the resource allocation problem developed in [6] considers the process of resource allocation as an optimal control problem. It uses Pontryagin's maximum principle to find the optimal allocation of energy between growth and reproduction for a given organism.

The idea of finding the optimal allocation of resources between growth and reproduction is the motivation behind this work. The aim is to develop a numerical method for the optimisation problem.

The work begins by looking at the optimal control problem and Pontryagin's Maximum principle before moving on to look at the biological problem and it's analytical solution in greater detail. The numerical method for this problem is then developed in chapter four. The method begins by solving the state and adjoint equations for an arbitrary control u, before looking at ways of making the control u optimal. The model developed is assessed by using trial problems to which analytical solutions can be found.

Pontryagin's principle was developed to deal with control problems where the variables were subject to magnitude constraints of the form:- $_i(\ )$ $_i$. This

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*

T

0

0

T

* * * * *

T

*

In time dependent problems Pontryagins principle can be used to determine either a minimum or a maximum of the functional J(u).

Consider the angular motion of a ship which is given by

$$\frac{\ ^2}{\ _2} + \frac{}{} = \qquad (2\ 7)$$

where the rudder setting u is subject to 1. Find u to minimise the time required to change course from $= 1$, $\frac{\partial\phi}{\partial t} = 0$ when $= 0$ to $= 0$, $\frac{\partial\phi}{\partial t} = 0$ when $=$. The first order equations are formed by taking $_1 =$, $_2 = \dot{}$. Then

$$\dot{} = \dot{}_1 = \ _2 \qquad _1(0) = 1 \qquad _1(\ ) = 0 \qquad (2\ 8)$$

$$\ddot{} = \dot{}_2 = \ _2 \qquad _2(0) = 0 \qquad _2(\ ) = 0 \qquad (2\ 9)$$

Hence we wish to

$$min \int_0^T 1 \qquad (2\ 10)$$

1 2 1 2 1 2 2 2

$$1 \quad \frac{}{1}$$

$$2 \quad \frac{}{2} \quad 2 \quad 1$$

1

2 t

where A and B are constants to be determined. Applying Pontryagins Principle we get

$$1 + \lambda_1^* x_2^* + \lambda_2^*(u^* - x_2^*) \leq 1 + \lambda_1^* x_2^* + \lambda_2^*(u - x_2^*) \qquad (2.16)$$

$$\lambda_2^*(u^* - u) \leq 0. \qquad (2.17)$$

Hence

$$u^* = -1 \quad if \quad \lambda_2 > 0 \qquad (2.18)$$

$$u^* = 1 \quad if \quad \lambda_2 < 0. \qquad (2.19)$$

From equation (2.15) we can see that λ_2 can change sign at most once. The equations for x_1 and x_2 are now found to be

$$x_1 = ut - Ce^{-t} + \qquad (2.20)$$

$$x_2 = u + Ce^{-t}, \qquad (2.21)$$

where C and D are constants to be found. There are two possible cases to consider: $u = 1$ at $t = 0$, $u = -1$ at $t = T$, and $u = -1$ at $t = 0$, $u = 1$ at $t = T$. From equations (2.15), (2.18) and (2.19) it can be determined that $u = 1$ at $t = 0$ is the correct case, x_1 and x_2 can now be found in terms of known values. Hence

$$x_1 = t + e^{-t} \quad for \quad t \leq t_s$$
$$x_1 = -t - e^{T-t} + T + 1 \quad for \quad t \geq t_s \qquad (2.22)$$

$$x_2 = 1 - e^{-t} \quad for \quad t \leq t_s$$
$$x_2 = -1 + e^{T-t} \quad for \quad t \geq t_s, \qquad (2.23)$$

from which we can determine the switching time t_s by equating the pairs of equations (2.22) and (2.23) for x_1 and x_2 at the switching time t_s, giving $t_s =$

$\frac{1}{2}(T-1)$. All that remains is to calculate the final time T. This is straight forward and gives $T = 2cosh^{-1}(e^{-\frac{1}{2}})$.

2.2 Singular Intervals

In the special case where the Hamiltonian H is linear in the control vector u, it is possible for the coefficient of the linear term to vanish over a finite interval of time. This gives rise to a singular interval and Pontryagin's principle gives no information about the control variable u. During this phase an additional condition is needed. The additional condition is a condition for local optimality, which holds through out the singular interval, and is the second order Clebsch-Legendre condition stated here without proof. The required condition for a maximum is

$$\frac{\partial}{\partial u}\left[\left(\frac{d}{dt}\right)\left(\frac{dH}{du}\right)\right] > 0 \tag{2.24}$$

For proof of this see [2], examples can also be found in this reference.

Example

Consider the control strategy that causes the response of the system

$$\dot{x_1}(t) = x_2(t), \qquad \dot{x_2}(t) = u(t) \tag{2.25}$$

subject to $x_1(0) = \alpha$, $x_2(0) = \beta$, to minimise

$$J = \frac{1}{2}\int_0^T \left(x_1^2(t) + x_2^2(t)\right)\, dt. \tag{2.26}$$

The final time T and the final states are free, and the controls are constrained by the inequality $\mid u(t) \mid \leq 1$. We now form the Hamiltonian function

$$H(x_1, x_2, u, t, \lambda_1, \lambda_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \lambda_1 x_2 + \lambda_2 u. \tag{2.27}$$

Applying Pontryagin's principle,

$$\frac{1}{2}\ {}^{*2}_{1} + \frac{1}{2}\ {}^{*2}_{2} + \ {}^{*}_{1}\ {}^{*}_{2} + \ {}^{*}_{2}\ {}^{*} \quad \frac{1}{2}\ {}^{*2}_{1} + \frac{1}{2}\ {}^{*2}_{2} + \ {}^{*}_{1}\ {}^{*}_{2} + \ {}^{*}_{2} \tag{2 28}$$

$$ {}^{*}_{2}(\ {}^{*} \quad) \quad 0 \tag{2 29}$$

Hence

$$ {}^{*} = 1 \qquad {}_{2} \quad 0$$

$$ {}^{*} = \quad 1 \qquad {}_{2} \quad 0 \tag{2.30}$$

Consider the existence of a singular interval $[\ _1\ \ _2]$, in which case $\ {}^{*}_{2} = 0$ for all

$[\ _1\ \ _2]$. Thus, for $[\ _1\ \ _2]$,

$$\dot{}_{2} = \ddot{}_{2} \qquad\qquad {}_{1}\ \ {}_{2}$$

$$ {}_{1} \qquad \frac{}{{}_{1}} \qquad {}_{1}$$

$$ {}_{2} \qquad \frac{}{{}_{2}} \qquad {}_{2} \qquad {}_{1}$$

$$ {}_{1}\ {}_{2}\ \ {}_{1}\ \ \ {}_{2}$$

$$ {}_{u=u^*}$$

$$ -\ {}^{2}_{1} \quad -\ {}^{2}_{2} \quad {}_{1}\ {}_{2} \quad {}_{2}$$

$$ {}_{1}\ {}_{2}$$

$$ -\ {}^{2}_{1} \quad -\ {}^{2}_{2} \quad {}_{1}\ {}_{2} \qquad {}_{2}$$

$$ -\ {}^{2}_{1} \quad -\ {}^{2}_{2}$$

$$ {}_{1}\ {}_{2}$$

$$ {}_{1} \quad {}_{2}$$

or

$$_1 \qquad _2 = 0 \qquad (2\ 37)$$

Differentiating equation (2.36) and substituting in the state equation gives

$$\dot{}_2 = \qquad _2 \qquad (2\ 38)$$

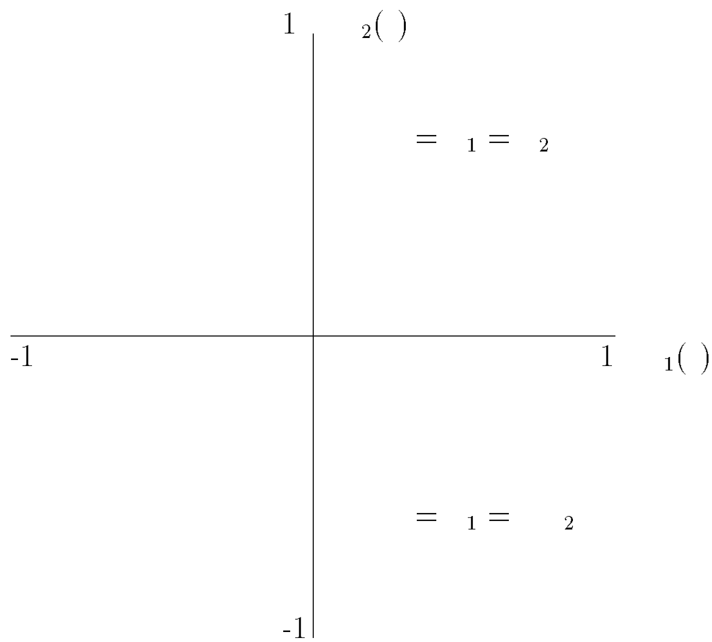
which implies

$$^*(\ ) = \qquad ^*_2(\ ) \qquad\qquad [\ _1 \quad _2] \qquad (2\ 39)$$

and, therefore, $^*_1(\ )$, $^*_2(\ )$ $[\ \ 1\ 1]$. Similarly, differentiating equation (2.37), it can be shown that

$$^*(\ ) = \ ^*_2(\ ) \qquad\qquad [\ _1 \quad _2] \qquad (2\ 40)$$

and $^*_1(\ )$, $^*_2(\ )$ $[\ \ 1\ 1]$. quations (2.36) and (2.37) define the locus of a point in the state plane ($_1$ $_2$) where singular intervalss may exsist. Since the system



moves away from the origin on the line $_1 = \ _2$, this segment cannot form part

2

2

2

2

*

whilst

$$\dot{}_2(\ _1) \quad 0 \qquad\qquad _1 + \ _2 \quad 0 \tag{2 49}$$

Comparing with equation (2.44), it follows that switching is only allowed in the case when $_1 + \ _2 \quad 0$. There are three possible cases:

1. $\frac{1}{2}\ ^2 + \frac{3}{2}$

2. $\frac{1}{2}\ ^2 + \frac{3}{2} \qquad \frac{1}{2}\ ^2 + 4$

3. $\frac{1}{2}\ ^2 + 4$

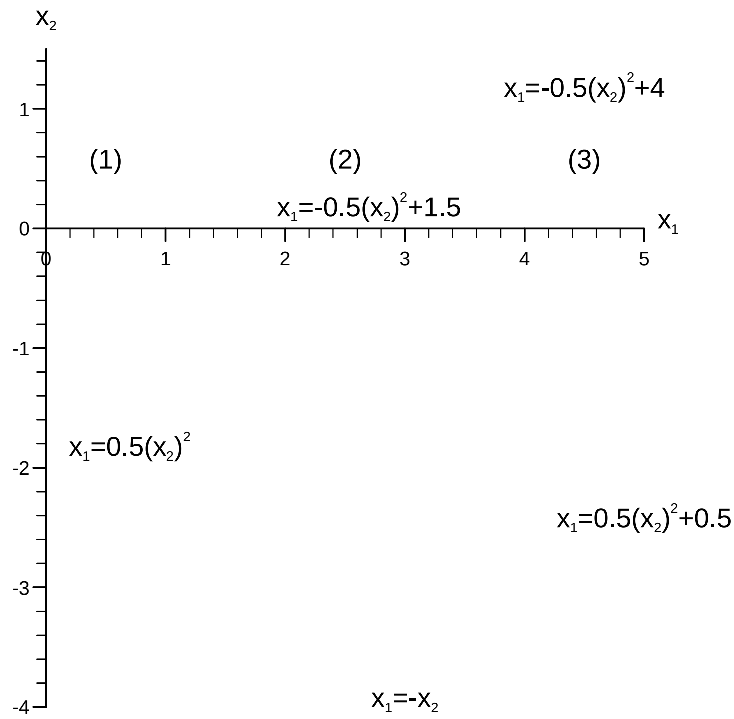
$_1 \quad _2$

$_1 \quad _2$

$_1 \quad \frac{1}{2}\ _2^2 \qquad \frac{1}{2}$

$_1 \quad \frac{1}{2}\ _2^2$

Possible controls three phase



= age

= final time

( ) = body weight, (0) = initial weight

( ) = total resources available for allocation, 0

$\frac{w(0)}{k}$

$- \int_0^t \mu(x)dx$

$-rt$

$-rt$

3.2 Biological Problem

The biological problem of resource allocation is considered as an optimal control problem , where fitness is maxismised by the choice of the control variable. Fitness of a life cycle strategy is measured by it's rate of increase r, defined by

$$1 = \int_0^T e^{-rt} l(t) b(t) \, dt, \tag{3.1}$$

where T is the maximum age of reproduction, [4]. A strategy which maximises fitness is used as this is the most probable result of evolution.

The resources $P(w)$ available to a given organism of size w are always subject to the competing demands of growth and reproduction. Since $L(t) = e^{-rt} l(t)$, it can be thought of as a single factor which weights reproduction in equation (3.1) and decreases with time at the rate $r + \mu(w)$. Reproduction decreases in value with time due to the population increasing at a rate of r and the decrease in survival probability by a rate of $\mu(w)$. u denotes the proportion of resources directed to reproduction, the remainder of the resources are directed to growth. Using the model of growth and reproduction developed in [6] we have an optimal control problem in which we choose u to maximise r subject to the following constraints.

$$\dot{w} = (1 - u(t))P(w), \quad w(0) = specified \tag{3.2}$$

$$\dot{L} = -(r + \mu(w))L, \quad L(0) = 1 \tag{3.3}$$

$$\dot{\theta} = \frac{uP(w)L}{w0}, \quad \theta(0) = 0 \quad \theta(T) = 1 \tag{3.4}$$

$$\dot{r} = 0 \tag{3.5}$$

with $0 \leq u \leq 1$. The dot $(\dot{})$ denotes differentiation with respect to time. quation

(3.1) can now be rewritten as

$$\int_0^T \frac{uP(w)}{w0} L dt = 1. \tag{3.6}$$

When maximising r directly using Pontryagin's principle we need to define the Hamltonian H and from this we get the equations for the adjoint variables by differentiating with respect to the state variables. H is defined as

$$H = \lambda_0 \frac{u(t)L(t)P(w)}{w0} + \lambda_1(1 - u(t))P(w) - \lambda_2(r + \mu(w))L(t). \tag{3.7}$$

λ_3 does not appear in the hamiltonian as $\dot{r} = 0$. The adjoint equations are given by

$$\dot{\lambda_0} = 0, \tag{3.8}$$

$$\dot{\lambda_1} = \lambda_2 m' L - \frac{u\lambda_0 LP'}{w0} - (1 - u(t))\lambda_1 P', \quad \lambda_1(T) = 0 \tag{3.9}$$

$$\dot{\lambda_2} = \lambda_2 m(w) - \frac{u\lambda_0 P(w)}{w0}, \quad \lambda_2(T) = 0 \tag{3.10}$$

$$\dot{\lambda_3} = \lambda_2 L(t), \quad \lambda_3(0) = 0, \quad \lambda_3(T) = 1. \tag{3.11}$$

We note that $m(w) = r + \mu(w)$ and $m'(w) = \mu'(w)$ where (') denotes differentiation with respect to w.

3.3 Two phase solution

In a two phase solution the control u switches instantly from zero to one. The switching point can be found by setting $\frac{\partial H}{\partial u} = 0$, which in this case gives

$$\frac{\partial H}{\partial u} = \frac{\lambda_0 LP}{w0} - \lambda_1 P = 0. \tag{3.12}$$

The control u is dependent on the relationship between $\frac{\lambda_0 L}{w0}$, and λ_1 giving

$$u^* = 0 \quad if \quad \frac{\lambda_0 L}{w0} < \lambda_1 \tag{3.13}$$

Now using equation (3.6) we get

$$1 = \int_{t_1}^{T} \frac{P(w_1)L(t)}{w0}\, dt = \frac{P(w_1)L_1}{w0} \int_{t_1}^{T} e^{-(r+\mu(w_1))(t-t_1)}\, dt =$$

$$\frac{P(w_1)L_1}{w0(r+\mu(w_1))} \left(1 - e^{-(r+\mu(w_1))(T-t_1)}\right). \tag{3.18}$$

This equation gives a relationship between r and w_1. This can be shown graphically and the optimal strategy determined from this by observation. This reduces the optimal control problem to a simple static problem of finding w_1 to maximise r. A necessary condition for optimality is that

$$\frac{dr}{dw_1} = \frac{\partial f}{\partial w_1} \Big/ \frac{\partial f}{\partial r} = 0. \tag{3.19}$$

Since we assume $\frac{\partial f}{\partial r}$ does not equal zero, we require $\frac{\partial f}{\partial w_1}$ to be equal to zero. Hence from equation (3.18) we obtain

$$m_1^{'} \frac{P(w_1)}{m_1}(T-t_1)e^{-m_1(T-t_1)} + \left(\frac{P(w_1)}{m_1}\right)^{'} \left(1 - e^{-m_1(T-t_1)}\right) - 1 = 0 \tag{3.20}$$

where $m_1 = r + \mu(w_1)$ and $m_1^{'} = \mu^{'}$. We now have three equations (3.16, 3.18, 3.20) and three unknowns r, w_1 and t_1 from which the exact solution can be determined.

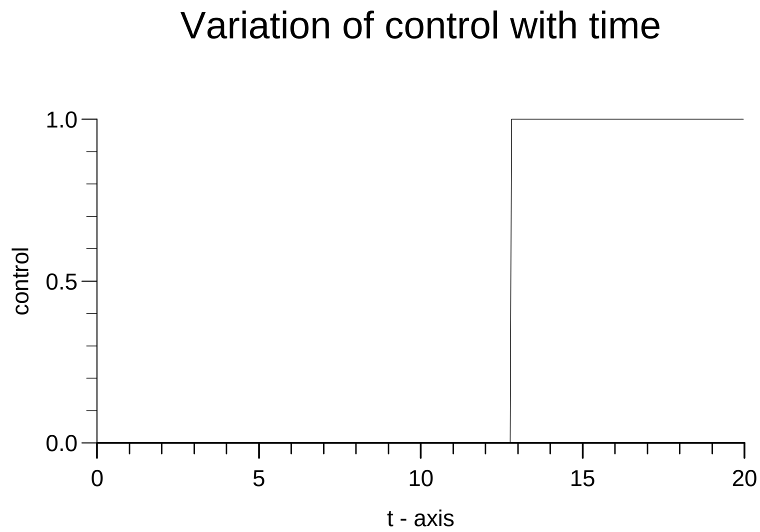


Figure 3.1: Two phase control

Example-linear

The functions are given as

$P(w) = 0.0702w$

$\mu(w) = 0.01w$

$w(0) = 0.25$

$k = 0.0602$

$T = 100$

$w0 = \frac{w(0)}{k}$

which on substitution into equations (3.16, 3.18, 3.20) give the following equations

$$w_1 = 0.25e^{0.0702t_1}, \tag{3.21}$$

$$w0 = \frac{0.0702w_1}{r + 0.01w_1}\left(e^{-\int_{w(0)}^{w_1}\frac{r+0.1w_1}{0.0702w_1}dw}\right)\left(1 - e^{-(r+0.01w_1)(100-t_1)}\right), \tag{3.22}$$

$$0.1\left(\frac{0.0702w_1}{r + 0.1w_1}\right)(100 - t_1)\,e^{-(r+0.01w_1)(100-t_1)} +$$

$$\left(\frac{P(w_1)}{r + 0.1w_1}\right)'\left(1 - e^{-(r+0.1w_1)(100-t_1)}\right) - 1 = 0. \tag{3.23}$$

1

1

$\frac{\lambda_0 L}{w0}$ 1

2

$'$

2 2

$\frac{P(w_2)}{r+\mu(w_2)}$ $'$

$\frac{P' \quad m}{'}$ $'$

2

$\frac{(\quad_2)}{+ \ (\quad_2)}$ $'$

$\frac{' \quad (\ + \ )}{'}$ $'$

$'$

2

A general three phase solution is looked at, in the simple case where the functions $P(w)$ and $\mu(w)$ are any general linear function. The first and second switching points are denoted by the subscripts 1 and 2 respectively. quation (3.16) relating w_1 and t_1 remains valid as does equation (3.17) relating w_1 and L_1. In the three phase optimal control problem, it is assumed that $\lambda_1 = \frac{\lambda_0 L}{w0}$ for a non- zero period of time. During this phase $\frac{\partial H}{\partial u} = 0$ and the trajectory is a singular arc. Since $\lambda_1 = \frac{\lambda_0 L}{w0}$ during the singular arc we can differentiate to get

$$\dot{\lambda_1} = \frac{\lambda_0 \dot{L}}{w0}. \tag{3.24}$$

Substituting equation (3.9) into this equation and using equation (3.3) as well as the relationship $\lambda_1 = \frac{\lambda_0 L}{w0}$ gives

$$\lambda_2 = \frac{\lambda_0(P' \quad m(w))}{w0m'}, \tag{3.25}$$

where ($'$) denotes differentiation with respect to w. λ_2 can now be found during the final phase from equation (3.10) giving

$$\lambda_2 = \quad 1 \quad e^{-m(w_2)(T-t_2)} \quad \frac{\lambda_0 P'(w_2)}{w0m(w_2)}, \tag{3.26}$$

which on substitution into equation (3.9) gives

$$\frac{\lambda_1 w0}{\lambda_0} = m_2' \frac{P(w_2)}{m_2} L(T)(T \quad t) + \quad \frac{P(w_2)}{m_2}' \quad [L(t) \quad L(T)]. \tag{3.27}$$

At the start of the final phase $\frac{\lambda_1 w0}{\lambda_0} = L(T)e^{m(w_2)(T-t_2)}$ which on substituting into equation (3.27) gives the criterion equation

$$m_2' \frac{P(w_2)}{m_2}(T \quad t_2)e^{-m_2(T-t_2)} + \quad \frac{P(w_2)}{m_2 w}' \quad 1 \quad {}^{-m_2(T-t_2)} \quad 1 = 0 \qquad (3\ 28)$$

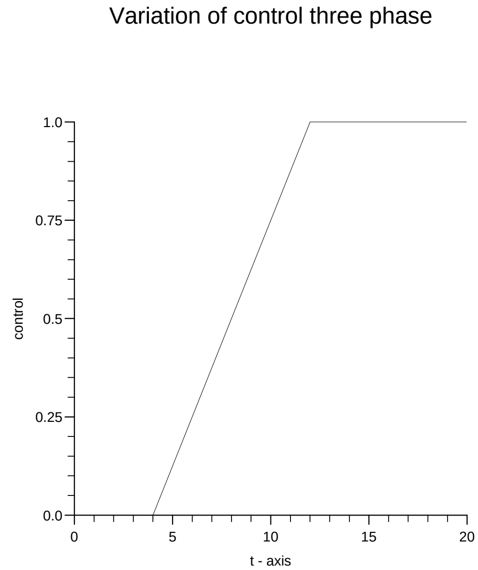


Figure 3.2: Three phase control

Example-linear

The functions are given as

$P(w) = 0.0702w$

$\mu(w) = 0.01w$

$w(0) = 0.25$

$k = 0.0602$

$T = 100$

$w0 = \frac{w(0)}{k}$

which on substitution into equations (3.16), (3.28), (3.29), (3.32), (3.33) give the following equations

$$w_1 = 0.25e^{0.0702t_1}, \tag{3.34}$$

$$0.1\left(\frac{0.0702w_1}{r+0.1w_1}\right)(100-t_1)\,e^{-(r+0.01w_1)(100-t_1)} +$$

$$\left(\frac{P(w_1)}{r+0.1w_1}\right)'\left(1-e^{-(r+0.1w_1)(100-t_1)}\right) - 1 = 0, \tag{3.35}$$

The development of the numerical method begins by deriving numerical schemes for solving the state equations for an arbitary control vector . The adjoint equations are then solved for this arbitary control vector using similar numerical methods. Since the equations for the adjoint variables are dependent on the state eI8Ne te to a final time $=$.

We commence with the state equation for , equation (3.2), which is independent. The remaining state equations are solved simultaneously. The function ( ) given in equation (3.2) is assumed to be problem specific and user defined. A simple trapezium rule discretisation is applied to equation (3.2) for , giving

$$_{j+1} = \ _j + \frac{\;}{2}((1 \quad _j) \ (\ _j) + (1 \quad _{j+1}) \ (\ _{j+1})) \qquad (4\ 1)$$

where is the step length ($=$ where is the final time and is the number of steps), and j denotes the number of steps taken from $= 0$.

It can be seen that $(\ _{j+1})$ cannot be evaluated as $_{j+1}$ is unknown and so this form of simple discretisation cannot be used. Instead it is used to form an iterative method, such that

$$^{n+1}_{j+1} = \ _j + \frac{\;}{2}(1 \quad _j) \ (\ _j)) + \frac{\;}{2}(1 \quad _{j+1}) \ (\ ^{n}_{j+1}) \qquad (4\ 2)$$

where $_0$ is known and where $^1_{j+1} = \ _j$ at each step.

The iteration process is stopped when $^{n+1}_{j+1} \quad ^{n}_{j+1}$ at each step. Having developed a numerical scheme to find , we must now show that the method converges. It is assumed that satisfies the Lipschitz condition.

$$(\ ) \qquad (\ ) \qquad (4\ 3)$$

Then for convergence it is necessary that

$$_{j+1} \quad ^{n+1}_{j+1} \qquad _{j+1} \quad ^{n}_{j+1} \qquad 1 \qquad (4\ 4)$$

Using equation (4.2) and (4.4), we obtained

$$_{j+1} \quad ^{n+1}_{j+1} \quad - \quad _{j+1} \quad _{j+1} \quad ^{n}_{j+1}$$

which from equation (4.3) gives

$$ {}_{j+1} \quad {}^{n+1}_{j+1} \quad \frac{}{2}(1 \quad {}_{j+1}) \quad {}_{j+1} \quad {}^{n}_{j+1} \tag{4 6}$$

Hence for convergence

$$ \frac{}{2}(1 \quad {}_{j+1}) \quad 1 \tag{4 7}$$

which gives a bound on the size of h as $\frac{2}{A}$ where A is the Lipshitz constant.

The Lipshitz constant A is dependent on the function ( ). If ( ) is a linear function such that ( ) = , then

$$ (\ ) \quad (\ ) \ = $$

$$ = \tag{4.8}$$

which gives the Lipshitz constant A as . If ( ) is a non-linear function then the Mean value theorem is used to determine A.

$$ (\ ) \quad (\ ) \quad {}'(\ )(\quad) $$

$$ {}'(\ ) \tag{4.9}$$

giving a bound on the Lipshitz constant A as A $\quad {}'(\ )$.

We now continue by developing a numerical scheme to solve the remaining

$$ {}_{j+1} \quad \frac{\frac{h}{2} \quad {}^{n} \qquad {}_{j}}{\frac{h}{2} \quad {}^{n} \qquad {}_{j+1}} \quad {}_{j} \quad 0 $$

$$ {}_{j+1} \quad {}_{j} \quad \frac{}{} \quad {}_{j} \quad {}_{j} \quad {}_{j} \quad {}_{j+1} \quad {}_{j+1} \quad {}_{j+1} \quad 0 $$

iterations for the value of r with two initial guesses and evaluating L and θ respectively for each guess. If the end condition $\theta(T) = 1$ is not satisfied, a Secant method is used to update the value of r, which is then used to recalculate the value of L and θ until the end condition on θ is satisfied. r is updated by

$$r^{n+1} = r^n - \frac{g(r^n)(r^n - r^{n-1})}{g(r^n) - g(r^{n-1})}, \tag{4.12}$$

stopping only when $r^{n+1} - r^n$ is small enough. The function $g(r^n)$ is defined to force the end condition $\theta(T) = 1$; hence

$$g(r^n) = \theta(T) - 1. \tag{4.13}$$

The solutions found for w, L, θ and r are now used in determining the numerical solutions of the adjoint equations. Due to the interdependence of the equations, the equations for λ_0, λ_2 and λ_3 are solved simultaneously.

4.2 Solving the adjoint equations

The adjoint equations are equations (3.8), (3.9), (3.10) and (3.11). To begin solving the adjoint equations f λ_0, λ_2 and λ_3 the end conditions are used.

The numerical solution is found using the same technique as that used for finding r, L and θ and so only the outline is given here. Two initial estimates of the constant λ_0 are obtained and equation (3.10) for λ_2 and equation (3.11) for λ_3 are solved for these values, with only the conditions at $t = T$ being used. The end condition on λ_3 for $t = 0$ is used in the secant update to give

$$\lambda_0^{n+1} = \lambda_0^n - \frac{\lambda_3^n(0)(\lambda_0^n - \lambda_0^{n-1})}{\lambda_3^n(0) - \lambda_3^{n-1}(0)} \tag{4.14}$$

quations (3.10) and (3.11) for λ_2 and λ_3 are again solved by using a trapezium discretisation, working backwards in time from the final time $t = T$, so that

$$\lambda_{2_j} = \frac{(1 - \frac{h}{2}(r + \mu(w_{j+1})))\lambda_{2_{j+1}} + \frac{h\lambda_0}{2w0}(u_j P(w_j) + u_{j+1} P(w_{j+1}))}{(1 + \frac{h}{2}(r + \mu(w_j))} \tag{4.15}$$

$$\lambda_{3_j} = \frac{-h}{2}(\lambda_{2_j} L_j + \lambda_{2_{j+1}} L_{j+1}) + \lambda_{3_{j+1}}. \tag{4.16}$$

It is now possible to solve equation (3.9) for λ_1; this is again done with a trapezium discretisation, stepping back in time from $t = T$ to $t = 0$. This gives

$$\lambda_{1_j} = \frac{(1 + \frac{h}{2}(1 - u_{j+1})P'(w_{j+1}))\lambda_{1_{j+1}}}{(1 - \frac{h}{2}(1 - u_j)P'(w_j))}$$

$$-\frac{h}{2}\frac{(\lambda_{2_j}\mu'(w_j)L_j + \lambda_{2_{j+1}}\mu'(w_{j+1})L_{j+1})}{(1 - \frac{h}{2}(1 - u_j)P'(w_j))}$$

$$+\frac{h\lambda_0}{2w0}\frac{(u_j L_j P'(w_j) + u_{j+1} L_{j+1} P'(w_{j+1}))}{(1 - \frac{h}{2}(1 - u_j)P'(w_j))} \tag{4.17}$$

from which λ_1 can be found.

4.3 Ways of finding optimal solutions

A numerical method for finding the optimal control u, and hence the optimal solutions to equations (3.2-3.5) and (3.8-3.11) is needed, assuming such an optimal exists. To optimise the value of the control vector u, starting from an arbitrary control vector, two methods have been tried, the projected gradient method, and the conditional gradient method. The basic algorithms were taken from [1] and adapted as necessary to suit this problem.

4.3.1 Projected gradient method

A basic outline to this method is shown in the flowchart in Fig 4.1 The new approximation to the optimal control u^* is chosen such that

$$u_{new} = u_{old} + step\frac{\partial H}{\partial u_{old}}, \tag{4.18}$$

where $step$ is the step size, and $\frac{\partial H}{\partial u_old}$ is specified by equation (3.12). If u_{new} is greater than one then it is set to the maximum value of one; similarly if u_{new} is less than zero then it is set to zero. If the value of r (which we wish to maximise) has not increased, then the step size is halved and the process of finding a new control u is repeated until r has increased in value. The adjoint variables are then evaluated for the new control, as is $\frac{\partial H}{\partial u_{new}}$, which is used to determine if the solution has converged to the optimal control u. A new variable $\tilde{u}$ is evaluated such that $\tilde{u}$ equals zero if $\frac{\partial H}{\partial u_{new}}$ is less than zero and one otherwise, where $\frac{\partial H}{\partial u_{new}}$is evaluated using equation (3.12). Convergence is then measured by the inner product

$$< \frac{\partial H}{\partial u}, (\tilde{u} - u) >, \tag{4.19}$$

which is determined numerically as

$$h \sum_{k=1}^{n} \frac{\partial H}{\partial u}\bigg|_{u_k} (\tilde{u_k} - u_k). \tag{4.20}$$

This maximises the first variationof the functional over all possible choices of u. If this is sufficiently small then the process is said to have converged and the optimal control u has been determined; otherwise the step size is set to one again and the whole process repeated until the optimal value of u is determined.

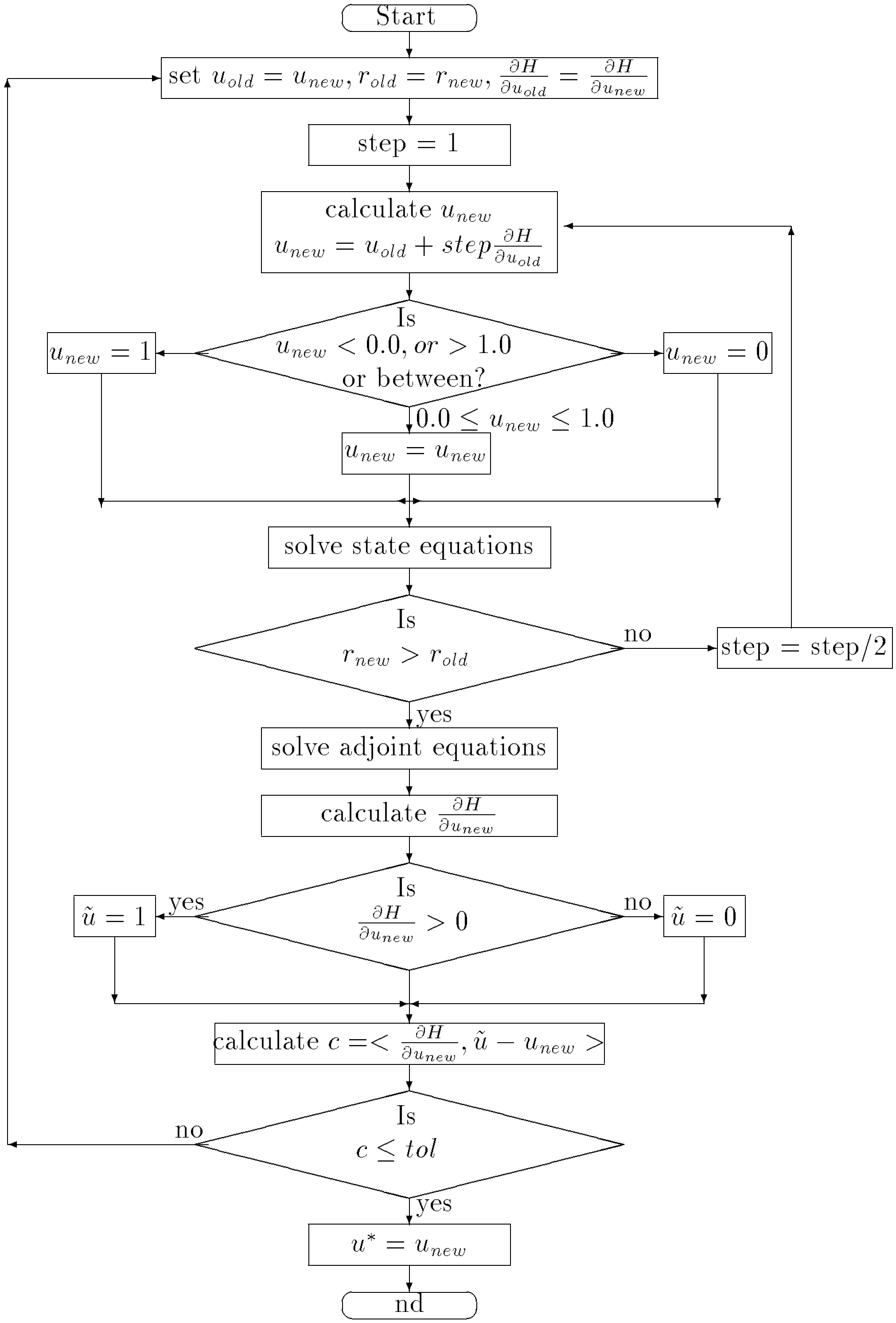


Figure 4.1: Projected Gradient Algorithim

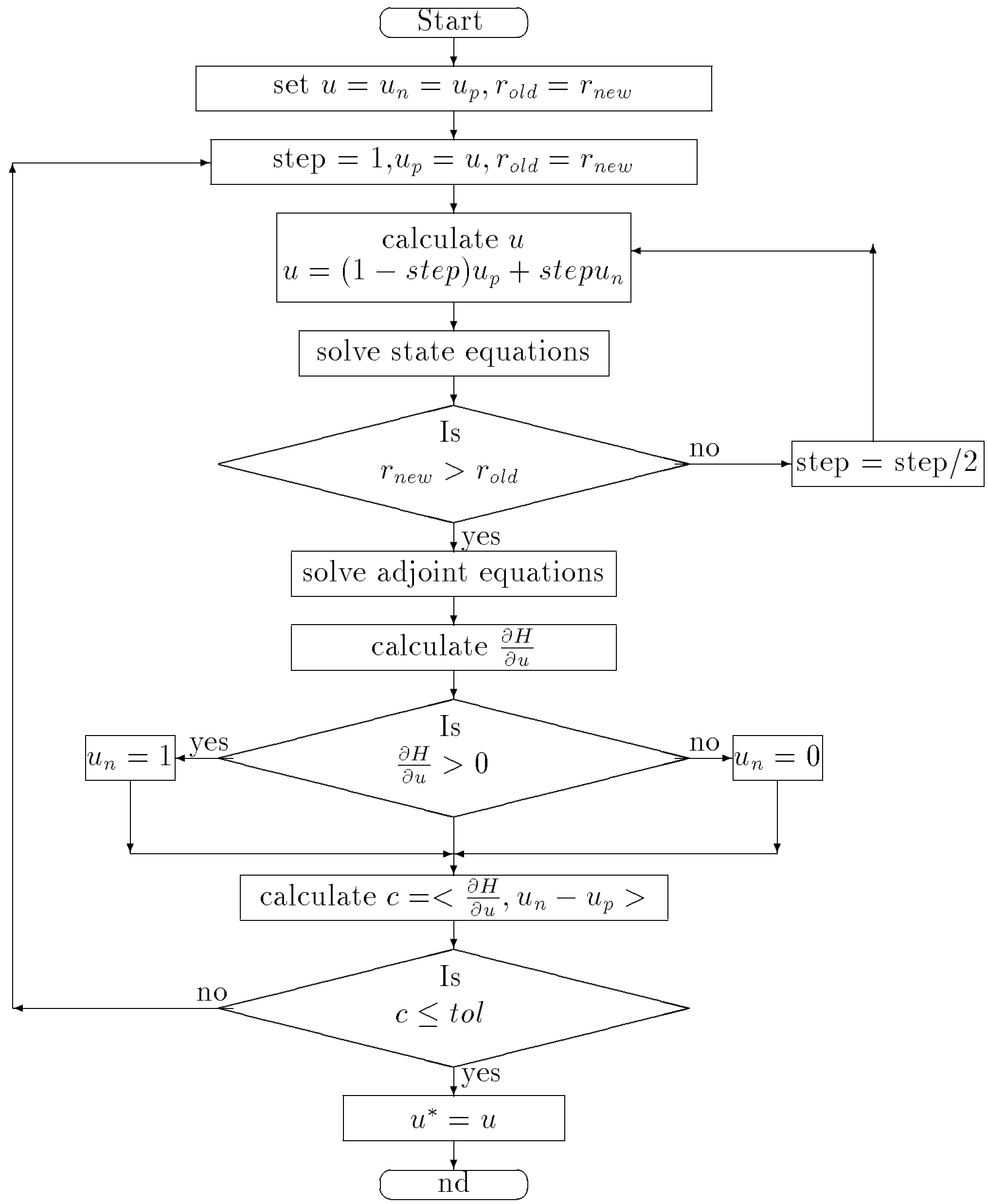


Figure 4.2: Conditional Gradient Algorithim

In this chapter the results from a series of test problems are presented. Due to the difficulty in calculating analytical solutions even in the case of linear functions for ( ) and ( ) the analytical solutions have only been fully calculated for a linear test problem. Tests are made which look at the sensitivity of the method to the initial approximation of , the effect on the solution of the initial arbitrary control and the effect of changing the length of the step taken in the gradient direction on the rate of convergence and accuracy of the solution for both optimisation are

i) (0), weight of at $= 0$

ii) $=$ final time

iii) The number of steps ($=$)

iv) Two initial approximations to the value of

v) Two initial approximations to the value of $_0$

vi) The switching point (zero to one) for the arbitrary initial control

which is not dependent on the optimisation method used. Points iv and vi are of primary interest.

The numerical method developed solves eight first order differential equations which are related in such away that seven of the eight differential equations are

$$_0 \quad _2 \qquad\qquad _0$$

$$_0 \quad _0 \int_0^T \frac{\quad}{\quad} \int_0^T \; _2 \quad '_2$$

$$_0 \int_0^T \; _2 \quad _2{}' \quad _2 \; _0^T$$

time $t = T$, given by

$P(w) = 0.0702w$

$\mu(w) = 0.01w$

$w(0) = 0.25$

$k = 0.0602$

$T = 100.$

The analytical solution has already been calculated (chapter 3) for this problem with both two and three phase solutions being found.

Projected Gradient Method

u switch point	no of iterations	r	$\lambda_0 = \lambda_2(0)$
t= 5	266	-0.0009	yes 0.0163
t=10	263	-0.0007	yes 0.0166
t=15	254	-0.0005	yes 0.0169
t=20	224	-0.0004	yes 0.0172
t=25	145	-0.0002	yes 0.0175
t=30	1	0.0	yes 0.0176
t=35	19	-0.0002	yes 0.0173
t=40	104	-0.0003	yes 0.0170

Table 5.2: The effect of moving the switching point for the Projected gradient method, $N = 2000$

The results in table 5.2 show that moving the switching point (t_s) has a quite dramatic effect on the number of iterations required for the method to converge. In all cases where a three phase solution is optimal, the projected gradient method

s

0

			0 2

s

inability to cope with problems that have a possible three phase solution containing a singular arc, which would render the method almost useless, or the magnitude of T causing some type of error propogation..

5.5 hanging the Step Length in the Gradient Direction

The step length in the gradient direction directly concerns the optimisation methods. The aim is to try and improve the rate of convergence without loosing accuracy. The projected gradient method only was considered in detail. When using the original step length of one the method appeared to be unusable, requiring over eight hundred iterations to converge to a solution when the switching point t_s was not very close to the exact solution. This caused errors to effect the solution resulting in a sub-optimal solution being found. Increases in the step length reduced the number of iterations required for convergence and the results are shown in table 5.4 for a variety os switching times as it would be most unusual for the initial guess to be almost identical to the exact solution.

It appears from the results in table 5.4 that very large increases in step length are possible without the solution being effected, while reducing the number of iterations required for convergence. However caution must be advised, as the step length possible is dependent on the size of the functional gradient, which for this example is very small. A step length of four would, however, be a good starting point for most problems. The development of a scaling method to determine the length of the step size based on the magnitude of the functional gradient would be a useful addition to the numerical method and is an area of future work.

[redacted]

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Conclusions

This project set out to develop a numerical method that was capable of solving the problem of finding optimal life cycle strategies for a given organism using the model developed in [6]. The numerical method developed here provides a solution to this problem.

The basic numerical method for solving differential equations (3.2-3.5) and (3.8-3.11) is simple in concept and design and could easily be extended to deal with further constraints in a more complex model. The two optimisation methods considered, the projected gradient and conditional gradient, have individual problems which need further consideration. The projected gradient method is very slow to converge if the arbitrary initial control has it's switching point away from the exact switching point. The conditional gradient method does not cope very well with problems whose solution contains a singular arc, either failing to converge or finding an alternative two phase solution if the switching point is chosen to be in almost the exact position.

The linear problem looked at in chapter five converges to a slightly different solution for each switching point specified using the projected gradient method with a maximum solution obtaining the value of $r = 0$ when the switching point is chosen to be almost the exact solution. There are a number of possible reasons why this may happen, it could be an inherent property of the problem caused by the exact solution containing an almost flat basin around the minimum causing the numerical method to converge to a series of sub-optimal solutions that are very close to the optimal as can be observed in table 5.2 and table 5.4. The most likely alternative is that the errors are a result of the convergence criteria not being strict enough on the outer iteration loop. The reason for not making these stricter lies in the number of iterations necessary for convergence being excessive if the tolerance used is less than 0.005.

It would be interesting to test the method with some real data and compare the strategies predicted by the numerical method and the observed strategies used by the organism. This would not only provide a means of validating the numerical method fully, but would also validate thoroghly the model developed in [6].

Acknowledgements

I would like to thank my project tutor Dr N. K. Nichols for her assistance with this work. I would also like to thank all other members of the mathematics department who have helped in the preparation of this work, and without who's help it would never have been completed. I thank S RC for their financial support.