# Flux Modelling of Polynyas


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Jonathan Muir


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JM

## Declaration

I con rm that this is my own work and the use of all material from other sources has been properly and fully referenced.

Signed . . . . . . . . . . . . . . . . . Date . . . . . . . . . . .

## Abstract

It is common in polynya flux models to take the depth of the pack ice, $H$, to be a constant value that needs to be predetermined before the polynya is modelled. This is undesirable because this depth influences the polynya width throughout the model time frame not just at the initial time, so by fixing its value beforehand we are losing some freedom in the model. There have been parameterisations that allow $H$ to vary with the frazil ice depth at the polynya edge, we take a different approach to reparameterising the depth on the polynya edge by looking at the pack ice as a sheet of ice which is allowed to diffuse and flow. We start by numerically and analytically evaluating the simple one-dimensional flux model for a latent heat polynya given by H.W. Ou in 1988. This flux model is then coupled to a diffusion PDE governing the pack ice. The diffusion PDE is given by the mass balance equation, with diffusion velocity obtained through Glen's flow law for ice. A depth profile for the pack ice is required for the pack ice as an initial condition. The coupled diffusion and polynya flux model is then modelled through a fixed computational grid and finally a moving grid method is applied to the model.

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# Chapter 1

# Polynya Introduction

A polynya is an opening of water enclosed in sea ice. Interactions between the atmosphere, ice and the ocean drive formation of polynyas. Polynyas are classified by the mechanisms that force their evolution; latent heat polynyas and sensible heat polynyas. Sensible heat polynyas are thermally driven by warm waters warming the ice and melting it to leave an area of water surrounded by pack ice, these most often occur out in open ocean waters. Latent Heat polynyas are driven by winds or ocean currents. We will be concerned with latent heat coastal polynyas for the remainder of the discussion.

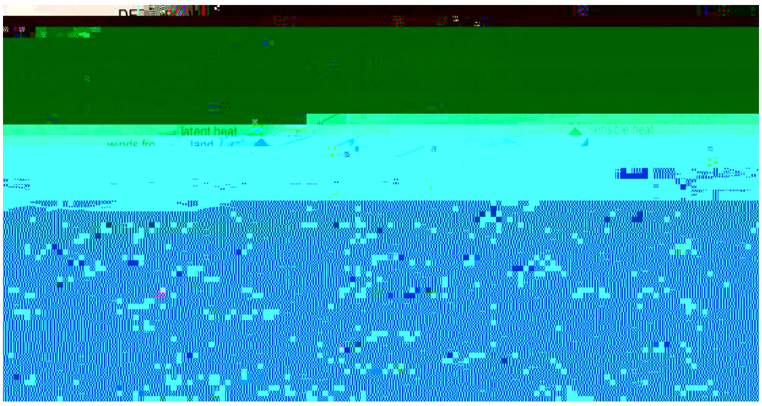

Figure 1.1: Schematic of Sensible Heat polynya and Latent Heat polynya [16]

The polynya opening is formed when some consolidated pack ice is forced by winds away from a coastline, or indeed from another block of rigid ice. Water within the polynya is cold enough that it is able to form ice plates that are suspended in the ocean [10], this ice is known as frazil ice. The production rate of frazil ice is affected by surface wind, air flows being warmed by the relatively warm polynya waters, downwind long wave radiation, evaporation of waters and the motion of tidal waters. This frazil ice produced within the polynya is forced out towards the pack ice by offshore winds.

The velocities of the frazil ice and the pack ice differ although both of these values are dependent on the wind velocity. The pack ice typically has a smaller velocity than that of the frazil ice, which results in there being a build up of frazil ice at the edge of the moving pack ice. This point where the polynya waters meets the pack ice is the polynya edge. Not only is there a difference in the speeds of the two types of ice, there is also a difference in directions that the ice moves. The pack ice is affected by the turning of the earth which is between 23 and 33 to the right (in the northern hemisphere) of the wind velocity [9]. The frazil ice is more susceptible to being affected by water circulations within the polynya waters, such as Langmuir circulations which are vortices that rotate in shallow waters and can vary the direction of the frazil ice flow up to 13 to the right of the wind velocity [9].

The evolution process of polynyas is divided into two phases, opening and closing. A coastal polynya opens when the forcing winds move the consolidated pack ice from the coastline, this will continue until either a steady state is reached or a change in forcing occurs. A steady state is where the polynya size has reached a maximum size and will not grow any further. The closing of a polynya is due to one of a couple of factors. Either there is a change in forcing which now forces the pack ice back to shore as well as the frazil ice in the region, or in *situ ice* formation within the polynya occurs during periods of slack wind, i.e. the polynya freezes over. In the first case the frazil ice will pile up on the coastal boundary instead of at the polynya edge as was the case during the opening of the polynya. [14]

Polynyas frequently occur in the arctic and antarctic over the colder winter months. The frequency of occurrence can vary, and there are instances where

Figure 2.1 shows this cross section that we are considering, . It is divided into three parts, (i) is the polynya - the area of water that is surrounded by sea ice or coastline, (ii) is the consolidated pack ice which is advected away from the coastline by prevailing winds and sea currents, and (iii) further out to sea we have the  rst year ice which is not included in the  ux models we consider below. The point of interest here is the boundary between the polynya and the consolidated ice. As the wind advects the ice away from the coastline, there is production of frazil ice in region (i) which is also advected away from the coast towards region (ii). The balance between the frazil ice production rate and its velocity, and the velocity of the pack ice is the driving idea behind the  ux models discussed below.

## 2.1 Lebedev, 1968

*Maximum Size of a Wind-Driven Lead During Sea Freezing*

Lebedev stipulated that the dimensions of a wind lead cannot grow inde -nitely [8], a lead is a fracture in the ice that is mechanically driven by the motion of ice of sheer forces, [10]. The base model he generates begins with an assumption that the heat loss in the production of ice has three main components.

rate of turbulent heat loss

$$q_1 = K_1 W(T_w \quad T_a)$$

rate of heat loss due to evaporation

$$q_2 = K_2 W(E \quad e) \ L$$

rate of radiation heat loss

$$q_3 = q_{eff}(1 \quad C) + \quad (q_d \quad q_s)(1$$

where ($K_{1;2}$) are proportionality constants, ($T_w; T_a$) are temperatures at the water and at height $a$ respectively, $W$ is wind speed, ($e; E$) is the pressure of the vapor at height $a$ and saturation vapor at temperature $T_w$ respectively, is the water density, ($q_d; q_s$) are direct and solar radiation values, and is the water albedo. $C$ is the degree of cloudiness on appropriate scales, $q_{eff}$ is e ective incident radiation with a clear sky at thermal equilibrium of air and water, ( ; ) are correction factors for e ective radiation.

These three factors are summed together to get an expression for the rate of change of the frazil ice. This is then integrated over a period of time to where a certain depth of ice is obtained. This time frame obtained for the ice depth to be reached can also be expressed in terms of the polynya width and the drift rate of the surface water and ice in the polynya. By combining these two expressions we get an equivalence for the size of the polynya and the change in ice depth. [8].

$$R = \frac{W \quad h}{K}$$

model that describes the evolution of the polynya edge $R$ as

$$RF = H \quad U \quad \frac{dR}{dt}$$

where $U$ is the advection rate of the consolidated ice, $F$ is the production rate of frazil ice, and $H$ is the depth of the consolidated ice pack. Pease notes that when $U; H;$ and $F$ are linear functions of position or approximately linear then we have a di erential equation with solution

$$R = \frac{UH}{F} \quad 1 \quad exp \qquad \frac{tF}{H} \qquad :$$

For large timescales we have a limit on the size of the polynya, bounded by $R = \frac{UH}{F}$. Pease also remarks that the expression for the maximum polynya size gives access to nding a timescale for the polynya to reach a given proportion of its maximum size, for example a time until the polynya reaches 95% of its maximum size. This timescale is only dependent on the freezing rate scaled by the collection thickness: for a greater freezing rate the time until maximum polynya size is shorter.

Pease gives a formulation for the freezing rate over the polynya. It takes into account the unre ective shortwave radiation, downward and upward long wave radiation, sensible/turbulent heat ux, and the latent heat of evaporation. The latent heat of evaporation is ignored since its contribution is negligible. Similarly the shortwave radiation contribution is zero during winter months and in the summer contribution is also negligible, so these e ects are neglected in the model. $F$ is the averaged area production rate and is equivalent to the local production rate $\frac{dH}{dt}$ [13].

$$F = \frac{e_a T_a^4 \qquad e_w T_w^4 + \quad {}_a C_h C_p V_a (T_a \quad T_w)}{{}_i L} \tag{2.1}$$

Here $U$ is a fraction of the wind speed, (normally 3% or 4%), is the Boltzmann constant, ${}_i$ is the new ice density, ${}_a$ is the cold air density, and $L$ is the latent heat of freezing for salt water. (All other constants/variables have been de ned as in Lebedev's paper.)

Pease initially tests the sensitivity of the model by investigating the effects that varying the surface temperature, $T_a$, the advection rate on the surface, $V_a$ and the maximum polynya size have on the model. The results were that the coastal polynyas reach a stable size within a typical synoptic time scale for low temperatures but not those approaching freezing. [13]

Three experiments in the Bering Sea during 1982, 1983 and 1985, were observed and they provide a means of testing the accuracy of the model. They don't cover a full range of variables contained in the model but were adequate in testing the mid ranges of the variables temperature and wind speed. The conclusion here is that for low wind speeds the model will not be useful since the idea of a collection thickness won't really come into physical being. Similarly for higher temperatures (those approaching freezing point from below) the model won't be appropriate due to lack of model physics [13].

## 2.3 Ou, 1988

*A Time-Dependent Model of a Coastal Polynya*

Pease's flux model assumes that the newly produced frazil ice in the polynya is instantaneously deposited at the polynya edge, this could be interprteted as the frazil ice having an infinite velocity towards the polynya edge. This is not the case in the physical world since infinite drift rate is not possible, the frazil ice has a finite drift rate. Ou extends the Pease model by accounting for the drift rate of the frazil ice. The equation for the depth of the frazil ice on the polynya edge is

$$h_t + (hu)_x = F$$

where $h$ is the frazil ice depth, $u$ is the frazil ice velocity, $F$ is the production rate given by (2.1), and the $x$ derivative is the change in position of the frazil ice with $x = 0$ denoting the coastline and $x = R$ denoting the polynya edge as with the Pease model. We have an initial coondition that when $x = 0$, $h = 0$. If we have a steady forcing term, i.e. the wind speed is constant and the frazil ice has constant force applied, then the depth of frazil ice at the polynya edge is

$$h_R = \frac{FR}{u_R} \tag{2.2}$$

where $u_R$ is the velocity of the frazil ice at the polynya edge, similarly $h_R$ denotes the frazil ice thickness at the polynya edge. This results in the flux balance

$$h_R\left(u_R - \frac{dR}{dt}\right) = H\left(U - \frac{dR}{dt}\right). \tag{2.3}$$

Ou's alteration to the Pease model keeps the property of having a steady state solution satisfying

$$R = \frac{HU}{F} \tag{2.4}$$

The model's sensitivity to small perturbations was examined and Ou concludes that high frequency variations in atmospheric conditions do not have a large impact on the polynya edge. By this Ou means that the period of the variations are small in comparison to the transit time of the frazil ice. Varying the ratio between the velocities of the consolidated ice and the frazil ice will affect the response at the polynya edge. Reducing the frazil ice drift speed (with respect to the consolidated ice speed) will lengthen transit time for the frazil ice, but will reduce the inertia of the system and prompt faster response of the polynya edge. Changes in air temperature will affect the production rate and are more effective in producing a response in the polynya edge, [11].

Ou recognises that the model is a very idealised version of the physics involved in the polynya process and ignores some important features namely;

- Although the frazil ice drift speed is assumed directly proportional to the wind speed, in reality we would expect a slight lag to occur between the two, i.e. the response to the wind velocity is not instantaneous

- Although the thickness of the consolidated pack ice, $H$, is taken to be constant (usually kept to be around $0.1m - 0.2m$) this is mainly due to lack of knowledge of the collection process at the polynya edge

Since spatial uniformity of the forcing terms has been assumed, it does not take into account strong wind gusts or regional variations across the polynya.

Ou's inclusion of the frazil ice drift results in a two-phase appearance to the process, an initial 'runaway' phase where the consolidated ice pack runs away, and then an 'approach' phase where the polynya edge reaches a steady state when the balance of  uxes reach an equilibrium [11]. When this equilibrium is reached the polynya has reached its maximum size, this is given by (2.4).

In Chapter 3 we look at the Ou model, taking a numerical and an analytical approach to modelling the polynya. Then in Chapter 4 we extend the model by including a new parameterisation for the pack ice.

# Chapter 3

# One Dimensional Flux Model

The model obtained by Ou shows the balance between the fluxes of the frazil ice and the consolidated pack ice at the polynya edge. This balance governs the position of the polynya edge. We also have that the frazil ice depth is given by a quotient of the production rate over the width of the polynya divided by the velocity of the frazil ice: equations (2.3) and (2.2). By rearranging the flux balance equation (2.3) we obtain

$$\frac{dR}{dt} = \frac{HU - h_R u_R}{H - h_R} \tag{3.1}$$

In the most simplified case we could have, we take $H$, $U$ and $u_R$ to be all known constants. Similarly, we can take $F$ to be constant by suitably fixing the constants within the parameterisation of the frazil ice production rate (2.1). Individual parameter values are given by Pease and shown in table 3.1, these

which is separated and integrated to get

$$\int dt = \int \frac{u_R H - F R}{u_R H U - u_R F R} dR.$$

We note that this is an implicit solution for $R$. Plotting this function taking a wind speed of $20ms^{-1}$, air temperature at $-20^{\circ}C$, and we use (2.1) for the production rate giving $0.29 \times 10^{-6} ms^{-1}$. The frazil ice and pack ice velocities are 3% and 2% of the wind speed respectively.

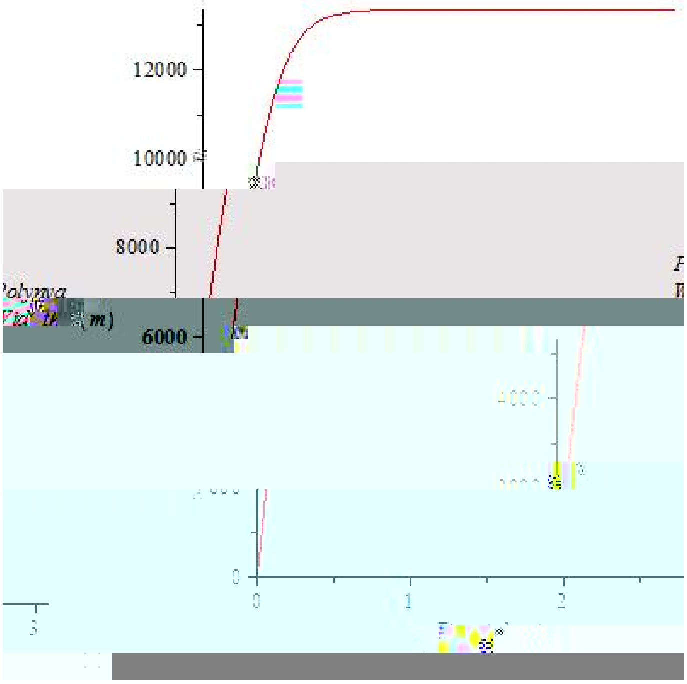


Figure 3.1: Analytical Solution to the steady state of a polynya opening using Ou's 1988 flux model with analytical solution (3.3)

We see that the width of the polynya will be approaching the steady state width (2.4) given to be

$$R = \frac{HU}{F} = 13357.52m.$$

The timeframe and the steady state width is what we expect from the literature, in Ou's concluding section of his paper he said the steady state was reached in less than two days [11]. Here our plot indicates that the steady state is being approached after just over a day from the initial forcing.

## 3.2 Numerical Solution

We use a Runge Kutta Fourth order scheme (RK4) to numerically integrate (3.1) and to determine the steady state polynya width. The algorithm for the RK4 is given below, the subscript, $i$, denoting the time at which we are calculating the position $R_i$ which in turn denotes the position of the polynya edge at time $t_i$.

$$
\begin{aligned}
R_{i+1} &= R_i + \frac{t}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad (3.4)\\
k_1 &= f(R_i; t_i)\\
k_2 &= f \quad t_i + \frac{t}{2}; R_i + \frac{k_1 \quad t}{2}\\
k_3 &= f
\end{aligned}
$$

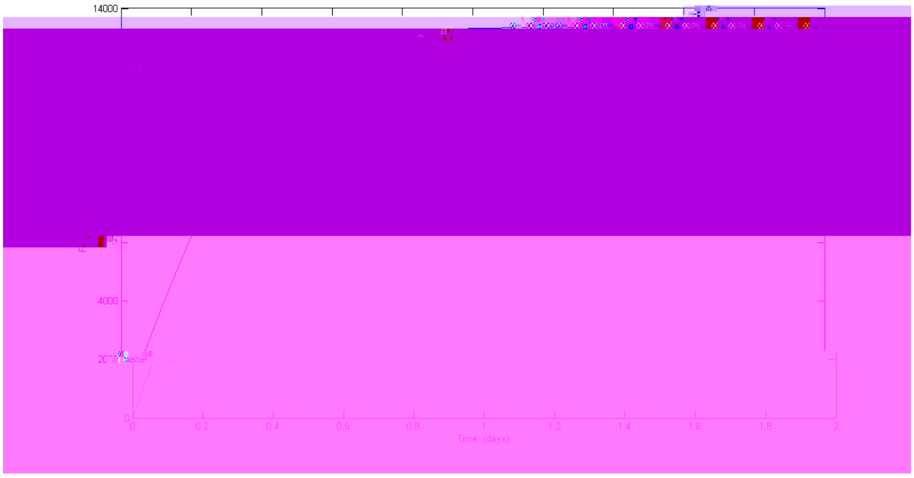

Figure 3.2: Runge-Kutta fourth order

This scheme is fourth order accurate, so that halving the time step should reduce the error between the approximation and the true solution by a factor of 16. We need to check that the solution we obtain for the RK4 method re ects this property. By taking a set time to integrate the RK4 scheme forward to and having a value that we keep as the true solution, successively halving the time step and taking the ratio of the di erence between the base solution and the time step solution we will see the a common ratio that illustrates the accuracy of the scheme.

| *t* | R (m) | Error | Ratio |
|---|---|---|---|
| 100 | 11462.8772215 | 0 | |
| 200 | 11462.8772211 | 0.0000004 | |
| 400 | 11462.8772153 | 0.0000062 | 0.06451633 |
| 800 | 11462.8771205 | 0.0000101 | 0.06138614 |

Table 3.2: Accuracy table for RK4

Table 3.2 shows the width of the polynya after 12 hours calculated using successively smaller time steps, the ratio in the nal column is roughly with a wind speed of $20ms^{-1}$.

Figure 3.3 shows by varying the wind speed the size of the steady state width is largely unaffected, see table 3.3 in relation to the steady state width (2.4)). The rate of convergence to this steady state is however affected. The steady state polynya width is in the region of $13.5km$, but by increasing the windspeed the time taken to reach the steady state width is altered.

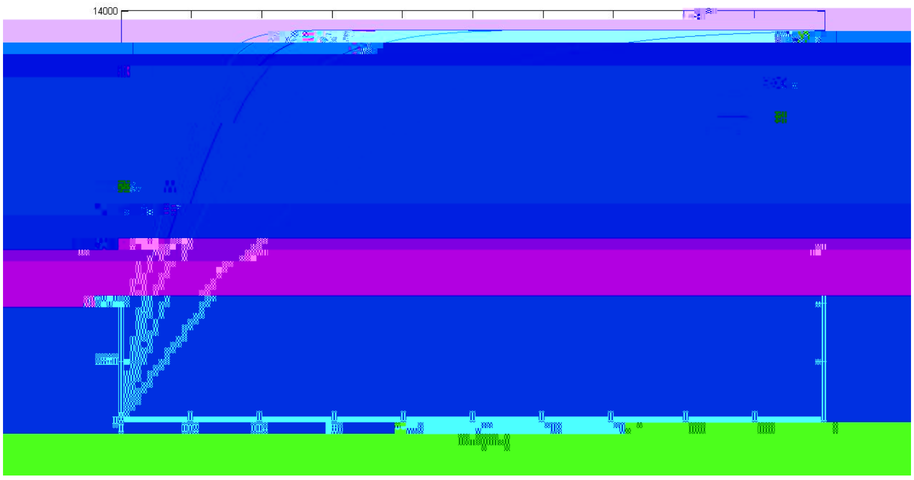


Figure 3.3: Runge-Kutta fourth order, showing the effect that varying the wind speed has on the polynya when temperature is kept constant at -20°C

By changing the air temperature in the production rate formula (2.1) we alter the steady state polynya width. Figure 3.4 illustrates this idea: lower air temperatures the size of the polynya is reduced as the production rate of the frazil ice increases, conversely for higher temperatures the steady state width of the polynya is increased. This would concurr with assessments Pease made about her model that the temperature has a greater influence than wind speed on the polynya response [13].

# Chapter 4

# Diffusion of Ice

So far we have kept the consolidated ice thickness constant at $0.1m$. In reality this is a big simplification as there are many factors that will alter the depth of the pack ice. The model we have considered thus far (3.1), has kept the value of $H$ constant. By allowing the frazil ice collection depth to evolve we can conceivably reach a point where depth of the frazil ice can exceed that of the consolidated pack ice, $h_R \geq H$. Problems arise in the current model because of a division by zero in (3.1). The motivation in changing the parameterisation of the collection thickness $H$ lies in trying to reduce the number of tunable parameters in the model used to model the polynya.

Some subsequent amendments to the model parameterise the collection thickness $H$ in such a way to keep the relation $h_R < H$, replacing $H$ by a function of $h_R, U$ and $u$. For example, in 2000 Biggs, Morales-Maqueda & Willmott [4] formulated a parameterisation of $H$ that would be dependent on the frazil ice depth at time $t$, and the relative velocities of frazil and pack ice at the polynya edge, satisfying

$$H = h_R + c(U - u)^2 \tag{4.1}$$

where $c = 0.665m^{-1}s^2$. The parameterisation was used in simulations of a St Lawrence Island polynya, alongside those that used the constant valued ice thickness. Comparing the results they found that they were similar to the simpler constant thickness parameterisation. St lawrence Island lies in the Berring sea to the west of Alaska, polynyas often form here in the winter months and thanks to its recurrent nature it has been used to verify results from various polynya models, [4, 9, 13]. A proposed edit of (4.1) was given in 2004 by Biggs and Willmott [5], and an additional additive term ($h_W$) was added to account for an effect of wave radiation stress

$$H = h_W + h_R + c(U - u)^2 \tag{4.2}$$

This was now a more robust formulation, referring to how this parameterisation discounts possibilities of division by zero in (3.1), that allowed for modelling of unsteady polynya opening.

The formulations above are derived in their respective papers for a two-dimensional model where the velocities of the frazil ice and the pack ice are able to travel in different directions to that of the wind direction. The simplification to a one-dimensional model is stated above so that there is no need for the formulae to have a component which forces the relative velocities to be evaluated in the normal direction to the polynya edge [16].

These parameterisations of the collection thickness are taken at one point, the polynya edge, and is only influenced by the frazil ice in the polynya waters. We now include the mass of the pack ice into a formulation of the ice thickness at the polynya edge by considering ice diffusion.

## 4.1 Ice Equation

### 4.1.1 Shallow Ice Approximation

The flux models we have seen so far have all used a simple flux function for the consolidated pack ice, $HU$. An alternative to this simple flux function is to look at the pack ice as an active block of ice that is allowed to flow. We know that the consolidated ice sheet is shallow compared to its width. This naturally leads to the application of a shallow ice approximation, including diffusion within the ice sheet. We apply the derivation of the mass balance and the diffusion model velocity forice flow equation, given by Partdidge and Baines [12], to the pack ice in our polynya system. This ice flow is achieved by using the mass balance equation

$$\frac{\partial H}{\partial t} + \frac{\partial (Hu)}{\partial x} = m \tag{4.3}$$

where $m$ is the depth averaged ice-equivalent accumulation rate, which is due to either thawing or additional ice from snow or freezing of water; however we will take $m = 0$ in our simple model. Here $u$ denotes the speed of ice diffusion which using Glen's Law for the flow of ice [7] leads to,

$$u = -cH^{n+1}H_x^n. \tag{4.4}$$

Here $c = \frac{2A\rho^n g^n}{n+2}$ is a constant when we assume that the density $\rho$ and temperatures are all constant. $A$ is a function of temperature which is given by Van der Veen in table 4.1 obtained through laboratory experiments and observations [15]

$$A = A_0 exp\left(-\frac{Q}{RT} + \frac{3C}{(T_r - T)^k}\right)$$

| Variable | Description | Value |
|---|---|---|
| $A_0$ | constant | $9.302 \times 10^7 kPa^{-3}yr^{-1}$ |
| $Q$ | activation energy for creep | $78.8kJ/mol$ |
| $R$ | gas constant | $8.321J/(molK)$ |
| $C$ | | $0.16612K^k$ |
| $T_r$ | 0°C in Kelvin | $273.39K$ |
| $k$ | | $1.17$ |

Table 4.1: Constants for temperature dependent constant within the flow rate given through Glen's Law

We will take $c$ to be a constant value as well and as such we assume that the ice pack has a spatially invariant temperature. The value of the ice temperature

which is as simple as taking the diffusion velocity away from the velocity given by the forcing wind we used in Chapter 3.

$$U_d = U - cH^5 H_x^3 \tag{4.8}$$

So using this form of the ice velocity we get the following polynya flux balance equation

$$\frac{dR}{dt} = \frac{HU - cH^5 H_x^3 - h_R u_R}{H - h_R}. \tag{4.9}$$

The replacement flux function is now more complicated than the previous one used, under the above assumption we have the flux function for the consolidated ice given by

$$H\left(\frac{dR}{dt} - U + H^4 H_x^3\right)$$

The boundary conditions are given in Section 4.1.1.

## 4.2 Steady State Solution

The inclusion of the diffusion term on the pack ice affects the maximum size that the polynya can reach. We expect that this width is now less than that of the simpler Ou model. The equation for the pack ice at the polynya edge is now

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\left(HU - cH^5\left(\frac{dH}{dx}\right)^3\right) = 0. \tag{4.10}$$

For a steady state the time variation of the ice depth is zero, so now

$$\frac{\partial}{\partial x}\left(HU - cH^5\left(\frac{dH}{dx}\right)^3\right) = 0$$

which we can integrate directly with respect to $x$ to find

$$HU - cH^5\left(\frac{dH}{dx}\right)^3 = constant. \tag{4.11}$$

We allow $x \to \infty$ to utilise our far field boundary condition within the ice pack which says that the spatial derivative is zero; $\frac{dH}{dx} = 0$. We are left with the constant being

$$H_\infty U \tag{4.12}$$

where the subscript $1$ denotes the pack ice thickness at the point where $\frac{dH}{dx} = 0$. This value of $H_1$ is assumed to be known from some source, either from satellite data, physical readings, or even knowledge of physical processes. However at this stage we do not know the exact values for $H_1$, so we choose a range of $H_1$ values and see how this parameter affects the steady state width.

Now (4.12) can be substituted into (4.11) forming a non-linear ODE,

$$\frac{dH}{dx} = \sqrt[3]{\frac{HU \quad H_1 U}{cH^5}}: \tag{4.13}$$

In order to calculate this steady state width for the diffusion model a shooting method will be used. First we set two values for the polynya width $R$, that will give us the initial values for integrating (4.13) forward using the RK4 scheme. These two values for $R$ will be an upper and lower limits that will be refined through an iterative loop. The initial values are obtained through

$$H = \frac{FR}{U}:$$

The boundary condition on the right hand side is given by the choice of $H_1$ We know what the depth of the frazil ice on the polynya edge will be given a certain polynya width $R$, so integrating forward from this point to find a corresponding $H_b$ value (the depth of the pack ice at its far field boundary) is relatively simple. From here we can adjust the polynya width used to find the frazil ice depth at the polynya edge.Then repeat the process iteratively until the $H_b$ value found lies within a certain distance from $H_1$, at this point we have the steady state polynya width which relates to the specified $H_1$.

It would not make sense to integrate backwards from the $H_1$ mark to get to the polynya edge because we would be integrating from an unknown point to a known point, integrating from a known condition to find an unknown variable is preferable.

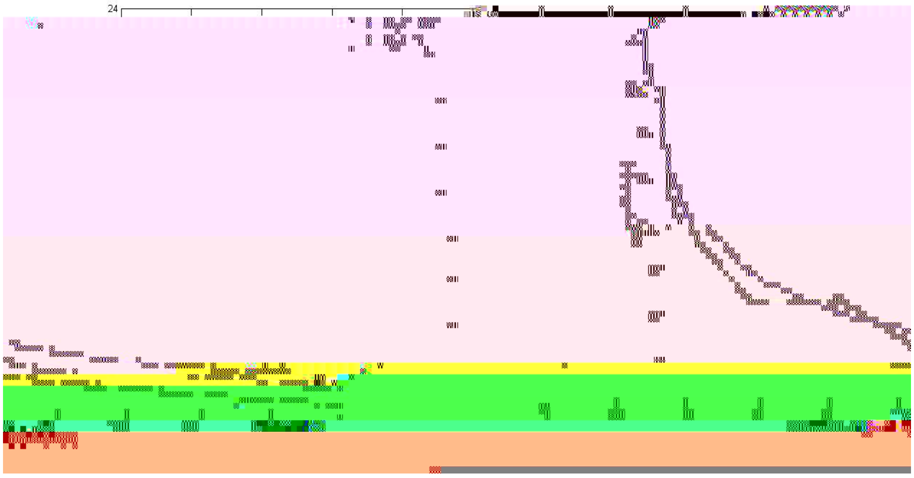


Figure 4.1: Steady state width with distance of $H_1$ from the polynya edge

The values used for the constants in this steady state method are wind speed is $20ms^{\ 1}$, air temperatuere is  20 C, and we use (2.1) for the production rate $F$. The frazil ice and pack ice velocities are 3% and 2% of the wind speed respectively. To see how the steady state is a ected we vary $H_1$ between $2m$ and $3m$. Also the distance between the polynya edge and the far eld boundary is varied between $1km$ and $200km$ to see how the length of the pack ice in uences the steady state. The results of using these values are given in Figure 4.1. We see that as we take the width of the pack ice to be larger then the steady state width approaches some limit from above. This is not surprising because the ice will be in a steady state, so in order for the ice thickness to be equal to the frazil

## 4.3 Numerical Approach

### 4.3.1 Ice Discretisation

To solve (4.7) numerically we discretise into an explicit finite difference approximation on a fixed computational grid $\{x_i\}$, as in Figure 4.2.

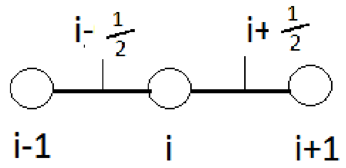


Figure 4.2: Spatial grid for the discretisation

The time derivative is discretised as

$$\frac{H_i^{k+1} - H_i^k}{t^{k+1} - t^k} \tag{4.14}$$

To discretise the spatial derivatives on the RHS of (4.7) we choose to evaluate the interior partial derivative of $H$ on halfway points between the nodes

$$\frac{\partial}{\partial x}\left(cH^5 H_x^3\right) = \frac{\partial(f)}{\partial x} \tag{4.15}$$

this can be deiscretised to

$$\frac{f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}}{\frac{1}{2}(x_{i+1} - x_{i-1})}.$$

now discretising the $f_{i+\frac{1}{2}}$ in 4.15 we have

$$\left(\frac{H_{i+1}^k + H_i^k}{2}\right)^5 \left(\frac{H_{i+1}^k - H_i^k}{x_{i+1}^k - x_i^k}\right)^3$$

similarly for the $f_{i-\frac{1}{2}}$

$$\left(\frac{H_i^k + H_{i-1}^k}{2}\right)^5 \left(\frac{H_i^k - H_{i-1}^k}{x_i^k - x_{i-1}^k}\right)^3$$

substituting these into (4.15) and putting it together with (4.14) we ghave the following discretisation

$$\frac{H_i^{k+1} - H_i^k}{t^{k+1} - t^k} = c\,\frac{\left(\frac{H_{i+1}^k + H_i^k}{2}\right)^5 \left(\frac{H_{i+1}^k - H_i^k}{x_{i+1}^k - x_i^k}\right)^3 - \left(\frac{H_i^k + H_{i-1}^k}{2}\right)^5 \left(\frac{H_i^k - H_{i-1}^k}{x_i^k - x_{i-1}^k}\right)^3}{\frac{1}{2}(x_{i+1} - x_{i-1})}. \tag{4.16}$$

The superscript, $k$, indicates the time discretisation and the subscript, $i$, denotes spatial discretisation.

Figure 4.3 shows di usion in a slab of ice 10m thick after 5000 time intervals of 100 seconds. The initial condition set on the slab of ice, although unrealistic it clearly shows the di usion process, is a sheer face at the left hand edge with an in nite gradient.

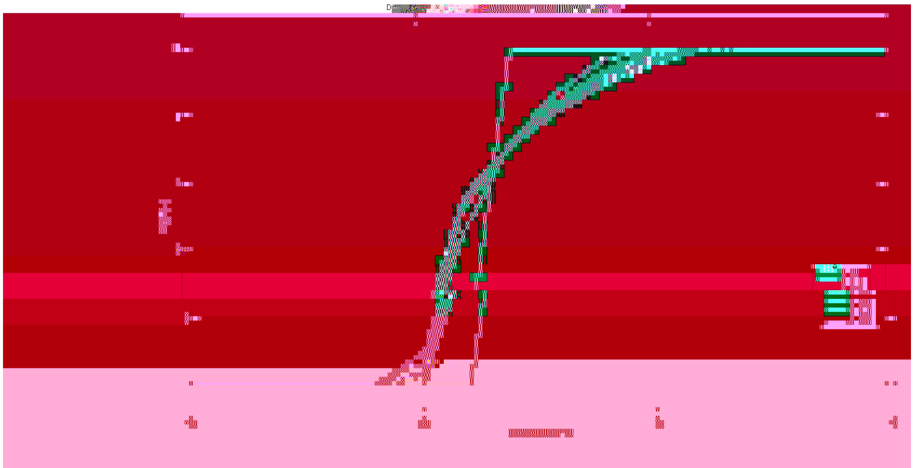

Figure 4.3: Di usion in a 10m slab of pack ice.

A more realistic shape for the initial conditions is illustrated in Figure 4.4. Here a $2m$

Figure 4.4: Di usion in a 2m slab of pack ice with a parabolic front

The di usion we see in the pack ice will be slower than shown in Figure 4.4, due to the thickness of the ice in the polynya model being less than a metre thick at the polynya edge. The two gures above show the expected form of the di usion but the rate of di usion will di er in the shallower ice. Also the boundary conditions we see at the polynya edge will mean that there will be a non zero thickness of consolidated ice at the polynya edge, whereas Figures 4.5 and 4.4 show a clear cut o where there is a zero ice thickness.

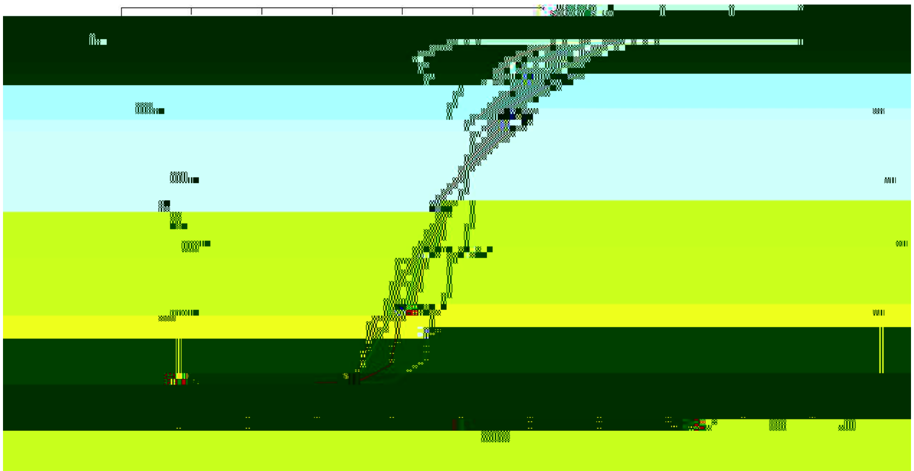

Figure 4.5: Di usion in a 1m slab of ice which has a tanh curve de ning the shape of the ice for its initial conditions

For Figure 4.4 a quadratic term depicts the front of the ice pack, but its shape is not continuous with the interior of the ice pack. A preferable initial condition is to have a scaled *tanh* curve for the ice depth throughout the pack ice at t = 0. This avoids there being discontinuities in the second derivative of the function at the join between the parabola and the constant line. This is shown in Figure 4.5. The di usion here is similar but without the hidden discontinuous rst derivative.

### 4.3.2 Polynya Model with Ice Discretisation

#### Polynya Di usion Model

In order to model the 1D polynya which incorporates the di usion equation for the pack ice as described in Section 4.1.2,oratesl7To1 cm q -436(pac(tocm qollo8(een436((des)28S(but)-

the ice pack will be allowed to di use using the discretisation (4.16). Then an averaged value over the rst few nodes of the ice pack will be used to model the polynya opening using the RK4 scheme (3.4). The reason for the averaging is to pass a depth value to the ux model (4.9) and without a local average on the polynya edge the depth would be very close to zero because the di usion velocity is very slow. This also allows us to evaluate the gradient of the ice at the boundary, which is required in the new velocity of the polynya edge. The new ux equation (4.9) will be used in the RK4 subroutine when the $k_{1;2;3;4}$ are called, which will advance the polynya edge forward.

This method has two way feedback between the polynya width and the diffusion of the pack ice The frazil ice depth, which is dependent on the polynya width, a ects the di usion in the form of boundary term on the polynya edge although it isn't a boundary it represents one because it occupies the arti cial domain of the computational grid. In turn the di usion rate a ects the width of the polynya by reducing the velocity of the pack ice from $U$ to $U_d = U \quad cH^5H_x^3$.

## Polynya Di usion Results

The parameters that we keep constant are the velocity of the frazil ice, $u_R$, the production rate, $F$, and the velocity of the pack ice $U$. In addition the constants in table 4.1 for the parameterisation of the di usion velocity prescribed through Glen's law for the ow of ice will also remain constant.

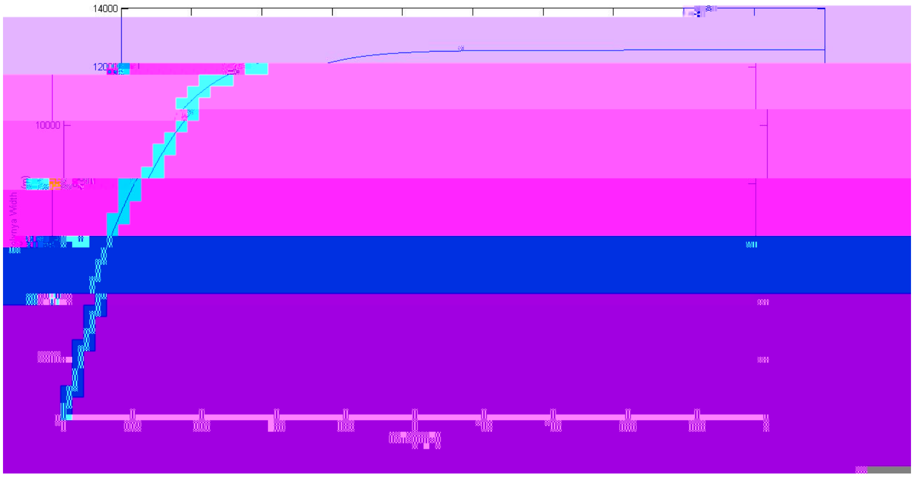

Figure 4.6: Polynya opening with di usion in the pack ice, wind speed is 20$ms^{1}$

Figure 4.6 shows that with the inclusion of the ice equation modelling di u-sion in the pack ice, the polynya opens up to a steady state, as expected. The time that the polynya takes to reach this steady state is similar to that in the simpler model with constant ice thickness, this time frame is between one and two days. The steady state width in this instance is just short of 12.5km. If we plot this against the polynya model with constant ice thickness we see how the ice di usion a ects the steady state width.

Chapter 5

# A Moving Mesh Method

and the consolidated ice. This is obtained by substituting the pack ice depth on the boundary given by (4.1) into the Ouflux balance equation (3.1). With an alteration to the (4.1) parameterisation to account for the diffusion velocity of the ice,

$$U_d = U - cH^5 H_x^3$$

On the right hand boundary of the pack ice we have the boundary conditions,

$$\begin{aligned} \frac{dH}{dx} &= 0 \\ \frac{dx}{dt} &= U. \end{aligned}$$

The total mass within the ice block is

$$\Phi(t) = \int_{a(t)}^{b(t)} H(x;t)\,dx \tag{5.2}$$

where a(t) and b(t) are the polynya edge and the far field boundary of the pack ice, respectively. Differentiating equation (5.2) with respect to time using Leibniz integral rule we have

$$\begin{aligned} \dot{\Phi} &= \frac{d}{dt}\left(\int_{a(t)}^{b(t)} H(x;t)\,dx\right) \\ &= \int_{a(t)}^{b(t)} \frac{dH}{dt}dx + \left[H\frac{dx}{dt}\right]_{a(t)}^{b(t)} \\ &= \int_{a(t)}^{b(t)} \frac{d}{dx}\left(cH^5\left(\frac{dH}{dx}\right)^3\right)dx + \left[H\frac{dx}{dt}\right]_{a(t)}^{b(t)} \\ &= \left[cH^5\left(\frac{dH}{dx}\right)^3 + H\frac{dx}{dt}\right]_{a(t)}^{b(t)} \\ &= H_{b(t)}U - \left[cH^5H_x^3 + H\left(\frac{h_R(U-u_r)}{0.665(U_d-u_R)^2} + U\right)\right]_{a(t)} \\ &= H_{b(t)}U - \left[cH^5H_x^3 + \frac{Hh_R(U-uR)}{0.665(U_d-u_R)^2} + HU\right]_{a(t)} \\ &= U\left(H_{b(t)} - H\right) - \left(cH^5H_x^3 + \frac{Hh_R(U-u_R)}{0.665(U_d-u_R)^2}\right) \\ &\neq 0 \end{aligned} \tag{5.3}$$

We drop the subscript $a(t)$ now and we have $H$ evaluated at the polynya edge unless otherwise stated.

We use a normalised conservation moving mesh principle, defining $x(t)$ by

$$\frac{1}{(t)} \int_{x(t)}^{b(t)} H\,dx = (x;b); \tag{5.4}$$

taken to be constant in time and therefore determined by the initial condition. is as defined in (5.3) so that if we have $x(t) = a(t)$ then $(a;b) = 1$. This $(x;b)$, represents a fraction of the mass that lies between the $x$ node and the far field boundary, it is constant in time.

Then, differentiating (5.4) with respect to $t$ we have

$$\frac{d}{dt}( (x;b)$$

This ensures that when we calculate the depth of the pack ice we keep the ratio of the mass in the cells to the total mass in the ice sheet constant. The depth ($H_i$) at the nodes ($X_i$) will be calculated after their positions have been found, the depths are found by using a discretisation of the conservation principle (5.4), in this case

$$H_i = \frac{^{k+1}}{^{0}} \frac{(x_i;b)}{(X_{i+1} \quad X_{i \quad 1})} \tag{5.8}$$

$^{0}$ represents the mass of the ice pack at the initial time. The ratio in the quantities scales the ratio to give the correct depth values such that the mass in the nodes remains at the same fraction of the total mass in the system.

The algorithm that we will follow to calculate the new position and depth of the ice at the nodes is as follows. First calculate the values for the $(x_i;b)$ constants at each node at time $t = 0$ using (5.4). Then calculate new positions (5.7) and finally we calculate the depth of the ice using the mid-point rule (5.8).

Using this method we model the polynya evolution by calculating the distance the first node of the ice pack has travelled, and it will also calculate the diffusion within the ice sheet. We have two way feedback in this model, just as was the case witgh the fixed grid model. The diffusion of the pack ice affects the rate of growth for the polynya, and in turn this affects the diffusion by means of the frazil ice depth providing conditions for the diffusion at the polynya edge.

## 5.1 Initial Moving Mesh Results

For the moving grid model we expect the polynya edge to be moving with the same behaviour as in the fixed grid model. We expect the diffusion in the pack ice to behave in a similar manner to the results seen in Figures 4.4,4.5.

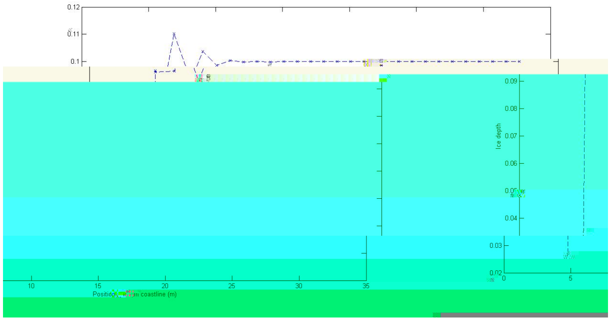

Figure 5.1: 12 second Movving grid instability

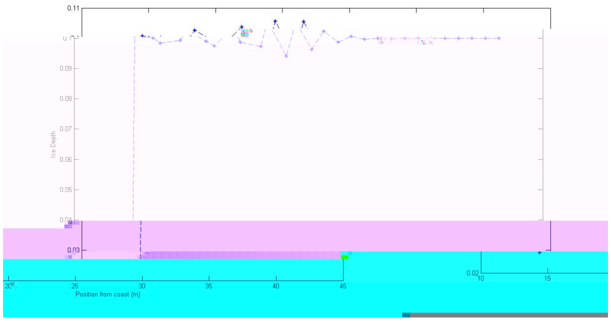

Figure 5.2: 36 second Movving grid instability

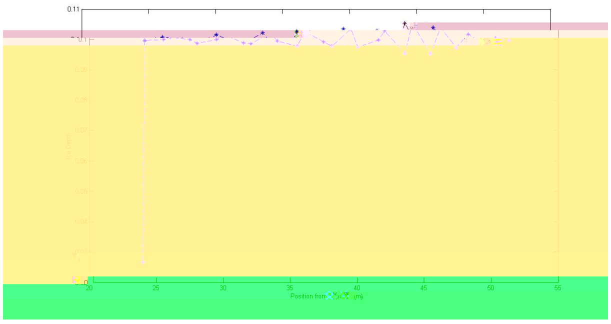

Figure 5.3: 60second Movving grid instability

These initial steps in the moving grid model show that there is instability in the di usion, with $t = 0{:}01s$ . This could be due to the discretisation grid being too coarse. This could be investigated by looking at the CFL condition, in order to control the size of these steps. The oscilations in the depth of the pack ice is quite noticable as the pack ice moves away from the coastline, Figures 5.1, 5.2, 5.3 show the increased oscoilations .


Figure 5.4: Initial sixty seconds of the moving grid model, illustrating the instability.

The behaviour of the polynya edge seems to be what we expect at this stage, the pack ice is moving away from the coastline at a rate which is the same as the ice pack velocity $0:4ms^{1}$ for a wind speed of $20ms^{1}$. The depth of the frazil ice is not great enough to slow the advancing polynya edge.

The oscilations in the pack ice are going to in uence the positions of the nodes, through 5.7 so the issue of the instability will need to be resolved before the steady state width reached. Neither are we going to see the correct di usion pro le of the ice until the oscilations at this early stage are resolved.

## 5.2 A change in Boundary Condition for the

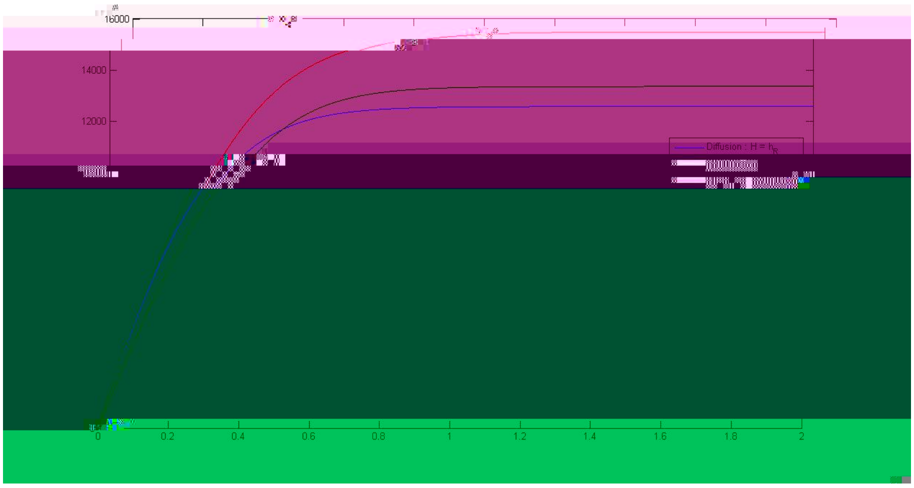


Figure 5.5: Comparative opening phase for a polynya with and without the altered boundary condition, wind speed is $20ms^{-1}$

Figure 5.5 illustrates how the new boundary condition for the di usion equation a ects the polynya's steady state, rather than the polynya opening to just under $12{:}5km$ the polynya opens to just over $15km$, which is not as close to the simple polynya ux solutions presented in Chapter 3.

# Chapter 6

# Conclusions

In Chapter 2 we reviewed the literature regarding one dimensional polynya ux models. The model presented by Ou in 1988 was then evaluated both analytically and numerically in Chapter 3d nraOu1 innalytform of the di usion: this is something that would need to be investigated in order to ensure that the results obtained correlate with the physical world. For(w)28(0ara1)implicitey9with modelled. Instead of taking the pack ice depth at the polynya edge, an averaged value for $H$ over a small area of the pack ice on the boundary was used, which avoided the division by zero that would have undoubtedly spoiled the results. This is not however a perfect fix for this problem since, by allowing a possibility of a division by zero the boundary condtion itself is flawed. Another issue also arose in that how far in to the pack ice do you take an averaged thickness for the depth on the polynya edge in order to run the polynya model.

The results from running the diffusive model with a fixed grid showed that the polynya evolution was slowed down and the steady state width was reduced by a couple of hundred meters. This is not unexpected since the diffusion will be acting in the opposite way to the motion of the polynya edge. In our diffusion model, width of the polynya is dependent the averaged depth of the pack ice at the polynya edge, this means that the size of the area that we take the average on affects the polynya width.

In Chapter 5 a different approach was used to model the polynya evolution. Rather than keeping the nodes in the numerical shceme fixed in place, the positions of the nodes were allowed to vary. The velocity for the moving mesh model was obtained through the new velocity of the pack ice at the polynya edge, $U_d$. By using a normalised conservation moving mesh principle we kept the mass in between the nodes to be time invariant. This approach doesn't look at the polynya edge primarily as the important piece of information: it is involved in calculating the position of the first node of the pack ice but the majority of the work involved modelled the diffusion of the ice on the moving grid.

The moving mesh model will need to be investigated further, since the results obtained from it are not what we would expect. The reasons for the unsatisfactory results could be from using a time step which is too large, this can be investigated by looking at the CFL condition for the model. The velocity of model, however with the instability in the depth calculations this needs to be resolved fully before a steady state is reached.

By considering diffusion in the pack ice, an alternative flux balance model was produced and used to model the evolution of a Latent heat coastal polynya to its steady state. The effect that the added diffusion parameterizatin had on this steady state width is hard to interpret because of the difficulties in having

# Chapter 7

# Further Work

In addition to investigating the current instabilities with the moving grid model there are natural extensions and further work that could be looked at, some of these are detailed below along with possible methods as to how they could be approached.

## 7.1 Seasonality

With the ice thickness parameterisation we could add in a seasonality term to add an effect that time of year has on the polynya modelling process. There are various ways we can do this to affect the diffusion rate. An additive constant dependent on time would be one way we could affect the diffusion. A scaled *sine* or *cosine* curve would be a sensible starting point to explore how the diffusion reacts, with the zeros on the curve indicating the start and end of winter and summer.

However the inclusion of the seasonality effect on the model may be largely unproductive since the lifespan of the opening phase of a coastal polynya is less than a few days and having a factor that changes due to effects that vary in term of months rather than days would not make a great deal of difference. Also since polynyas are predominantly a winter phenomenon.

Including a variable to account for weather patterns could be included to allow for the effects that a snow storm or warmer weather would have on the ice equivalent accumulation rate in equation (4.3). This would affect the temperature of the ice in the constant affecting the diffusion rate in Glen's Law (4.4).

## 7.2 Application to polynya closing models

Thus far the models we have been concerned with are all modelling the polynya opening to a steady state, we could apply the ice equation (4.5) to polynya closing models such as the one presented by Teal, Willmott, Biggs & Morales Maqueda [14]. This could be done in a similar manner to how we have done with the opening models. In this case we would expect the diffusion to aid the process of closing the polynya.

## 7.3 Extension to Two Dimensions

In the One-Dimensional models we considered a cross sectional view of the polynyas, but we could extend the diffusion term added to the pack ice into models which look at the two-dimensional polynya process.

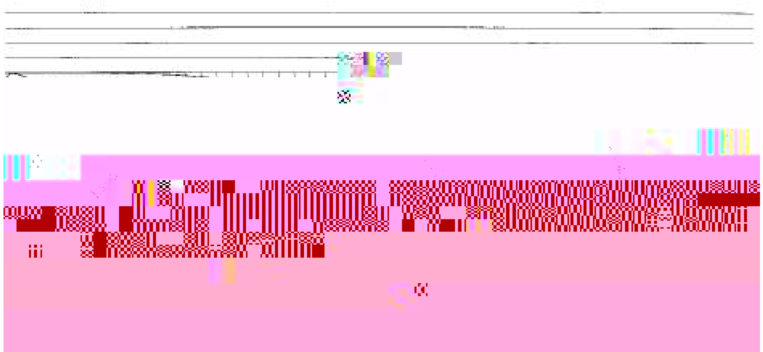

Figure 7.1: Two-dimensional polynya model diagram, adapted from [9]

This would allow us to look at the shape and the area of the polynya rather than the width at one point. A 2D time dependent flux model is presented in 2000 by Morales-Maqueda & Willmott (MMW) [9] that requires specification of the coastal boundary, time varying surface wind, shortwave radiation, air temperature and relative humidity. This is developed from previous two dimensional models given by Darby, Willmott and Somerville in 1995 [6] and Willmott, Morales Maqueda and Darby in 1997 [17]. This is a generalisation of the 1D model set out in the 1988 Ou paper: the generalisation to two dimensions given by

$$\nabla C \cdot \frac{H\mathbf{U} - h_C \mathbf{u}_C}{H - h_C} + \frac{\partial C}{\partial t} = 0$$

Now $\mathbf{U} = (U,V)$ is the velocity of the pack ice, and $\mathbf{u} = (u,v)$ is the velocity of the frazil ice. C is the curve $C(\mathbf{R},t)$ where $\mathbf{R} = (X,Y)$, is the curve of the

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