School of Mathematics, Meteorology
and Physics

Department of Mathematics

Preprint MPS_2010-01

11 January 2010

A high frequency boundary element method for scattering by convex polygons with impedance boundary conditions

by

S.N. Chandler-Wilde, S. Langdon and
M. Mokgolele

A high frequency boundary element method for scattering by convex polygons with impedance boundary conditions

S. N. Chandler-Wilde[1], S. Langdon[1,*], and M. Mokgolele[1]

[1] *Department of Mathematics, University of Reading, Whiteknights PO Box 220, Reading RG6 6AX, U.K.*

Abstract. We consider scattering of a time harmonic incident plane wave by a convex polygon with piecewise constant impedance boundary conditions. Standard finite or boundary element methods require the number of degrees of freedom to grow at least linearly with respect to the frequency of the incident wave in order to maintain accuracy. Extending earlier work by Chandler-Wilde and Langdon for the sound soft problem, we propose a novel Galerkin boundary element method, with the approximation space consisting of the products of plane waves with piecewise polynomials supported on a graded mesh with smaller elements closer to the corners of the polygon. Theoretical analysis and numericalespeult-261(ansugge)-301(antha-301(anthe26ter)mber)-331(an)-with respect to the frequency of the incident wave.

AMS subject classifications: 35J05, 65N38, 65R20

Key words: boundary integral equation method, high frequency scattering, convex polygons, impedance boundary conditions

---

1 Introduction

In this paper we consider two-dimensional scattering of a time-harmonic incident plane wave $u^i(\mathbf{x}) = \mathrm{e}^{ik\mathbf{x}\cdot\mathbf{d}}$

well-defined as bounded linear operators, see [15], where also our various function space notations are defined.) Then the scattering problem we consider is: given $\beta \in L^\infty(\Gamma)$, find the total field $u^t \in C^2(D) \cap H^1_{\mathrm{loc}}(D)$ such that

$$\Delta u^t + k^2 u^t = 0 \quad \text{in} \quad D, \tag{1.1}$$

$$\partial_{\mathbf{n}}^+ u^t + \mathrm{i}k\beta\gamma^+ u^t = 0 \quad \text{on} \quad \Gamma, \tag{1.2}$$

and such that the scattered field $u^s := u^t - u^i$ satisfies the Sommerfeld radiation condition

$$\frac{\partial u^s}{\partial r}(\mathbf{x}) - \mathrm{i}ku^s(\mathbf{x}) = o\left(r^{-1/2}\right), \tag{1.3}$$

as $r := |\mathbf{x}| \to \infty$, uniformly with respect to $\mathbf{x}$

The approach we will take in this paper combines ideas from our earlier work for sound soft convex polygons [6] with ideas developed for solving a two-dimensional problem of high frequency scattering by an inhomogeneous half-plane of piecewise constant impedance [7, 14]. In [7] a method in the spirit of the geometrical theory of diffraction was applied to obtain a representation of the solution, with the known leading order behaviour being subtracted off, leaving only the remaining scattered field due to the discontinuities in the impedance boundary conditions to be approximated. This diffracted field was expressed as a product of oscillatory and non-oscillatory functions, with a rigorous error analysis, supported by numerical experiments, demonstrating that the number of degrees of freedom required to maintain accuracy as $k \to \infty$ grows only logarithmically with respect to k. This approach was improved in [14], where derivation of sharper regularity estimates regarding the rate of decay of the scattered field away from impedance discontinuities led to error estimates independent of k.

The plan of this paper is as follows. In §2 we derive regularity results, demonstrating in particular that $\gamma^+ u^t$ can be written as the known leading order physical optics solution plus the products of plane waves with unknown functions that are non-oscillatory, highly peaked near the corners of the polygon and rapidly decaying away from the corners. In §3, we discuss the boundary integral equation formulation of (1.1)–(1.3). We describe our approximation space and Galerkin boundary element method in §4, and present numerical results demonstrating the efficiency of our scheme at high frequencies in §5. Finally, in §6 we present some conclusions.

2 Regularity results

Our aim in this section is to investigate the regularity of u^t, deriving bounds on derivatives which are sufficiently explicit, in particular in their dependence on the wavenumber, so that we can prove the effectiveness of our novel boundary element approximation space. In this endeavour we will, as part of our arguments, relate all bounds on derivatives to

$$M := \sup_{x \in D} |u^t(x)|. \tag{2.1}$$

We note first of all that $\gamma^+ u^t \in H^{1/2}(\Gamma) \subset L^2(\Gamma)$ so that the impedance boundary condition (1.2) implies that $\partial_{\mathbf{n}}^+ u^t \in L^2(\Gamma)$. It follows from standard regularity results for elliptic problems in Lipschitz domains [15, Theorem 4.24] that $\gamma^+ u^t \in H^1(\Gamma)$ and thus, from Theorem 6.12 and the accompanying discussion in [15], that $u^t \in H^{3/2}_{\mathrm{loc}}(D)$, so that, by standard Sobolev imbedding theorems [15], $u^t \in C(\bar{D})$ (as a consequence of which $M < \infty$).

From this point on in the paper we restrict attention to the case shown in figure 1 where Ω is a convex polygon. We write the boundary of the polygon as $\Gamma = \cup_{j=1}^{n} \Gamma_j$, where Γ_j, $j = 1, \ldots, n$, are the n sides of the polygon, with j increasing anticlockwise as shown in figure 1. We denote the corners of the polygon by P_j, $j = 1, \ldots, n$, and we set $P_{n+1} = P_1$, so that, for $j = 1, \ldots, n$, Γ_j is the line joining P_j with P_{j+1}. We denote the length

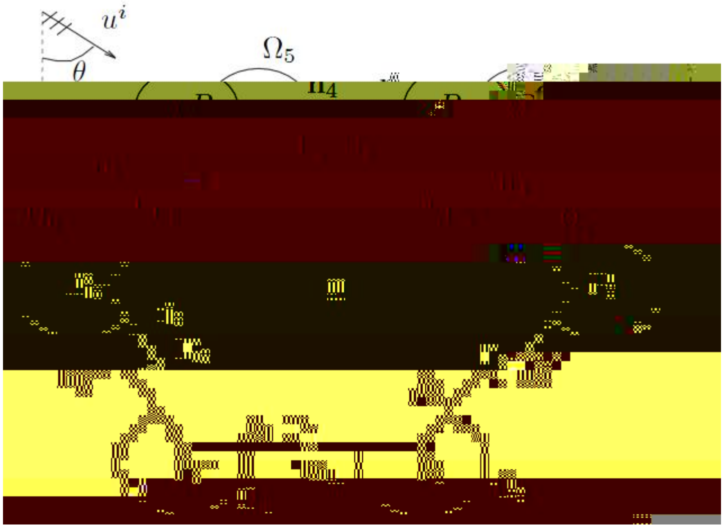

Figure 1: Notation for scattering by an impedance polygon

of G_j by $L_j = |P_{j+1} - P_j|$, the external angle at vertex P_j by $\mathrm{W}_j \in (p, 2p)$, and the outwards unit normal vector to G_j by $\mathbf{n}_j$. We let $q \in [0, 2p)$ denote the angle of the incident plane wave direction $\mathbf{d}$, as measured anticlockwise from the downward vertical $(0, -1)$. We also assume from this point on that b takes a constant value on each side of the polygon G_j; that is, $b(\mathbf{x}) = b_j$, $\mathbf{x} \in \mathrm{G}_j$

Taking the limit as $\mathbf{x} \to g_d$ we see that this equation holds also for $\mathbf{x} \in g_d$. Now the second and third integrals in (2.3) are continuously differentiable in $S_d \cup g_d$, with gradient whose magnitude is $\leq CM^*$ in $\overline{S_{3d/4}}$, where $M^* = \sup_{\mathbf{y} \in S_d} |u(\mathbf{y})|$ and C depends only on N and d. It follows from (2.3) (with $\mathbf{x} \in g_d$) and mapping properties of the single-layer potential (e.g. [9]) that $u \in C^{0,\#}(g_{pd})$, for every $p \in (0,1)$, with $\|u\|_{C^{0,\#}(g_{3d/4})} \leq CM^*$, where C here depends on all of $\#$, N, and d. Again using (2.3) and standard mapping properties of the single-layer potential [9], we see that $u \in C^1(S_d \cup g_d)$ and that the bound (2.2) holds. $\square$

Our first bounds on derivatives of u^t will be bounds on ∇u^t in D. We note first of all that it follows from standard interior elliptic regularity estimates [11, Theorem 3.9, Lemma 4.1] that there exists an absolute constant $C > 0$ such that, for every $e > 0$,

$$|\nabla u^t(\mathbf{x})| \leq Ce^{-1}(1 + (ke)^2)\, M \tag{2.4}$$

if $\mathbf{x} \in D$ and the distance of $\mathbf{x}$ from G, $\mathrm{dist}(\mathbf{x},\mathrm{G}) > e$. Using Lemma 2.1 applied to u^t in domains $\{\mathbf{x} \in D : |\mathbf{x} - \mathbf{x}^*| < 2c/k\}$ with $\mathbf{x}^* \in \mathrm{G}$, $c > 0$, and $\mathrm{dist}(\mathbf{x}^*, \{P_1, \ldots, P_n\}) > 2c/k$

a separation of variables solution in polar coordinates local to the corner P_j (cf. [6, Theorem 2.3]). The bound (2.6) is also consistent with the well-known Malyuzhinets solution (see [17] and the references therein) for scattering by a wedge with impedance boundary conditions.

In order to deduce more detailed regularity estimates we combine ideas from [6] (for the related sound soft problem) and [7, 14] (for an impedance half-plane problem). These more detailed estimates are bounds on derivatives of all orders related to the trace of the total field $\gamma^+ u^t$, relevant to the analysis of boundary element methods based on a direct integral equation formulation obtained from Green's theorem (see §3 below).

For $j = 1,\ldots,n$, let D_j denote the half-plane to one side of Γ_j given by $D_j := \{\mathbf{x} \in \mathbb{R}^2$

We note that the integral on the right-hand side is well-defined since u^t is continuous and bounded in $\bar{D}$ and ∇u^t satisfies the bounds (2.5) and (2.6), while $G_j(\mathbf{x},\cdot)$ decreases sufficiently rapidly at infinity [7, (2.10)] so that it is absolutely integrable on ∂D_j. We note also that, since the left and right hand sides of (2.8) are both continuous in $\overline{D_j}$, (2.8) holds in fact for all $\mathbf{x}\in\overline{D_j}$.

In the case that Γ_j is not illuminated by the incident wave (by this we mean the case that $\mathbf{d}\cdot\mathbf{n}_j\geq 0$), it can be shown that (2.8) holds for $\mathbf{x}\in\overline{D_j}$ also with u^s replaced by u^i. (The point is that (see [8, Remark 2.15, Theorem 2.19(ii)]), in the case $\mathbf{d}\cdot\mathbf{n}_j\geq 0$, u^i can be approximated in $\overline{D_j}$ by a bounded sequence of solutions of the Helmholtz equation which satisfy the Sommerfeld radiation condition and which converge uniformly on compact subsets of $\overline{D_j}$ to u^i, so that (2.8) holds first for each member of this sequence and then, in the limit, also for u^i.) Adding the equations (2.8) satisfied by u^s and u^i, we see that

$$u^t(\mathbf{x}) = -\int_{\Gamma_j^-\cup\Gamma_j^+} G_j(\mathbf{x},\mathbf{y})\left(\frac{\partial u^t}{\partial\mathbf{n}}(\mathbf{y})+\mathrm{i}k\beta_j u^t(\mathbf{y})\right)\mathrm{d}s(\mathbf{y}),\quad \mathbf{x}\in\overline{D_j}, \tag{2.9}$$

if Γ_j is a shadow side, since $u^t=u^i+u^s$ and $\frac{\partial u^t}{\partial\mathbf{n}}+\mathrm{i}k\beta_j u^t=0$ on Γ_j. On illuminated sides (where $\mathbf{d}\cdot\mathbf{n}_j<0$), (2.8) still holds for u^s and we can follow the argument of [7, p.653] to deduce that

$$u^t(\mathbf{x}) = u_j^t(\mathbf{x}) - \int_{\Gamma_j^-\cup\Gamma_j^+} G_j(\mathbf{x},\mathbf{y})\left(\frac{\partial u^t}{\partial\mathbf{n}}(\mathbf{y})+\mathrm{i}k\beta_j u^t(\mathbf{y})\right)\mathrm{d}s(\mathbf{y}),\quad \mathbf{x}\in\overline{D_j}, \tag{2.10}$$

where $u_j^t(\mathbf{x}):=u^i(\mathbf{x})+R_{\beta_j}(\theta-\theta_{j+1})u^r(\mathbf{x})$, u^r is the plane wave $u^r(\mathbf{x}):=\exp(\mathrm{i}k(c_j+\mathbf{x}\cdot\mathbf{d}_j'))$, with $\mathbf{d}_j':=\mathbf{d}\cdot\mathbf{s}_j\mathbf{s}_j-\mathbf{d}\cdot\mathbf{n}_j\mathbf{n}_j$ and $c_j:=P_j\cdot(\mathbf{d}-\mathbf{d}_j')$, and R_{β_j}

and

$$v_j^{-}(s) := -\int_{L_j}^{\infty} \check{k}_j(k(t-L_j+s))\mathrm{e}^{\mathrm{i}kt} g_j(t)$$

it is easily seen that $A_1(s) \leq C_m(c+s)^{-1/2-m}$, for $s \geq 0$, and that $A_2(s) \leq C_m s^{-1/2-m}$ for $s \geq c$, while, for $0 < s < c$,

$$
\begin{aligned}
A_2(s) &\leq C_m \int_{-c}^{0} (s-t)^{-m} |t|^{\alpha_j^+ - 1} \mathrm{d}t \\
&= C_m \left(\int_s^c (s+t)^{-m} t^{\alpha_j^+ - 1} \mathrm{d}t + \int_0^s (s+t)^{-m} t^{\alpha_j^+ - 1} \mathrm{d}t \right) \\
&\leq C_m \left(\int_s^c t^{\alpha_j^+ - 1 - m} \mathrm{d}t + s^{-m} \int_0^s t^{\alpha_j^+ - 1} \mathrm{d}t \right) \leq C_m s^{\alpha_j^+ - m}.
\end{aligned}
$$

From these bounds the bound (2.13) on $|v_j^{+(m)}|$ for $m \in \mathbb{N}$ follows. □

3 Boundary integral equation formulation

Where Φ is defined by (2.7), applying Green's representation theorem [15] to u^s gives

$$u^s(\mathbf{x}) = \int_\Gamma \left(\frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} \gamma^+ u^s(\mathbf{y}) - \Phi(\mathbf{x},\mathbf{y}) \partial_{\mathbf{n}}^+ u^s(\mathbf{y}) \right) \mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in D. \tag{3.1}$$

Applying Green's second theorem [15] to $\Phi(\mathbf{x},\cdot)$ and u^i in Ω we see that

$$0 = \int_\Gamma \left(\frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} u^i(\mathbf{y}) - \Phi(\mathbf{x},\mathbf{y}) \frac{\partial u^i}{\partial \mathbf{n}}(\mathbf{y}) \right) \mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in D. \tag{3.2}$$

Then adding (3.1) and (3.2) and using the boundary condition (1.2), we find that

$$u^t(\mathbf{x}) = u^i(\mathbf{x}) + \int_\Gamma \left(\frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} + \mathrm{i}k\beta(\mathbf{y}) \Phi(\mathbf{x},\mathbf{y}) \right) \gamma^+ u^t(\mathbf{y}) \mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in D. \tag{3.3}$$

Applying the trace operator γ^+ and using the jump relations [15, Theorem 6.11], we obtain a standard boundary integral equation (cf. [9, Section 3.9]) for $\gamma^+ u^t$, that

$$\frac{1}{2}\gamma^+ u^t(\mathbf{x}) = u^i(\mathbf{x}) + \int_\Gamma \left(\frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} + \mathrm{i}k\beta(\mathbf{y}) \Phi(\mathbf{x},\mathbf{y}) \right) \gamma^+ u^t(\mathbf{y}) \mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in \Gamma \setminus \{P_1, \ldots, P_n\}. \tag{3.4}$$

It is well known [9, 20] that, while (3.4) is uniquely solvable for all but a countable set of positive wavenumbers k, with the associated linear operator bounded and invertible on $H^s(\Gamma)$, for $0 \leq s \leq 1$, in particular on $L^2(\Gamma)$, (3.4) is not uniquely solvable for all wavenumbers. Precisely, if k is such that the Dirichlet problem for the Helmholtz equation in the interior region Ω has a non-trivial solution u_D (k is a so-called *irregular frequency*), then (3.4) has infinitely many solutions.To avoid this problem, the standard solution is to use a combined-layer formulation [2, 9], taking a linear combination of (3.4) with the equation that we get by applying the normal derivative operator $\partial_{\mathbf{n}}^+$ to (3.3),

and $f(s) = u^i(\mathbf{x}(s))$. The first step in our numerical method is to separate off the explicitly known high frequency leading order behaviour which we denote by $Y(s)$. From (2.11) and Theorem 2.1 it is clear that this leading order behaviour is

$$Y(s) := \begin{cases} u_j^t(\mathbf{x}(s)) & \text{on illuminated sides} \\ 0 & \text{on shadow sides.} \end{cases}$$

Introducing the new unknown $j = f - Y$, and substituting into (4.1), we have

$$j(s) - 2\int_0^L K(s,t)\, j(t)\,\mathrm{d}t = 2f(s) - Y(s) + 2\int_0^L K(s,t)Y(t)\,\mathrm{d}t, \tag{4.2}$$

which we can write in operator form as

$$(I - \mathcal{K})\, j = F, \tag{4.3}$$

where $\mathcal{K}v(s) := 2\int_0^L K(s,t)v(t)\,\mathrm{d}t$, $F(s) := 2f(s) - Y(s) + 2\int_0^L K(s,t)Y(t)\,\mathrm{d}t$, and I is the identity operator. Thinking of (4.3) as an operator equation on $L^2(0,L)$, this is the equation that we are going to solve for the unknown j by a Galerkin boundary element method.

We now design our Galerkin approximation space $V_{N,n} \subset L^2(0,L)$ in such a way as to efficiently represent j, based on the representation (2.11) and the bounds in Theorem 2.1 (note that the notations in (2.11) and (4.2) are related by

$$j(\tilde{L}_{j-1} + s) = f_j(s) - y_j(s), \quad \text{for } 0 \le s \le L_j,\ j = 1,\dots,n). \tag{4.4}$$

Our estimates in Theorem 2.1 are similar to those for the same scattering problem but with sound-soft boundary conditions [6, Theorem 3.3, Corollary 3.4], but with different exponents for $0 < ks \le 1$. Hence our approximation space is similar to (although not the same as) that defined in [6]. To describe this approximation space we begin by defining a composite graded mesh on a finite interval $[0,A]$, which comprises a polynomial grading near 0 and a geometric grading on the rest of the interval $[0,A]$. This mesh will be a component in the boundary element mesh that we will use on each side of the polygon.

Definition 4.1. For $A > l > 0$, $q \ge 1$, $N = 2,3,\dots$, we define $N_1 := \lceil Nq \rceil$ and $N_2$673 Tf 8.487 0 Td [(()]TJ/F98 10.9

on $L^2(a,b)$, $\|g\|_{2,(a,b)} := \left\{\int_a^b |g(s)|^2 \mathrm{d}s\right\}^{1/2}$. For $A > l > 0$, $n \in \mathrm{N} \cup \{0\}$, $q \geq 1$, where y_i, $i = 0,1,...,N_1 + N_2$, are the points of the mesh in Definition 4.1, let $\mathrm{P}_{N,n} \subset L^2(0,A)$ denote the set of piecewise polynomials

$$\mathrm{P}_{N,n} := \{s \in L^2(0,A) : s|_{(y_{j-1},y_j)} \text{ is a polynomial of degree } \leq n \text{ for } j = 1,\ldots,N_1 + N_2\},$$

and let P_N be the orthogonal projection operator from $L^2(0,A)$ to $\mathrm{P}_{N,n}$, so that setting $p = P_N g$ minimises $\|g - p\|_{2,(0,A)}$ over all $p \in \mathrm{P}_{N,n}$. Our error estimates for our boundary element method approximation space are based on the following theorem (cf. [6, Theorem 4.2]). We omit the proof which is a minor variant of the proof of [6, Theorem 4.2], referring the reader to [16] for details. Note that the relevance of this result is that, by Theorem 2.1, $v_j^{\pm}$ satisfies the conditions of this theorem with $a = a_j^{\pm}$.

Theorem 4.1. *Suppose that $g \in C^{\infty}(0,\infty)$, $k > 0$, $A > l := 2p/k$ and $a \in [\frac{1}{2},1]$, and that there exist constants $c_m > 0$, $m = 0,1,2,...$, such that, for $m \in \mathrm{N}$,*

$$|g(s)| \leq \begin{cases} c_0, & 0 < ks \leq 1, \\ c_0 (ks)^{-1/2}, & ks \geq 1, \end{cases} \qquad |g^{(m)}(s)| \leq \begin{cases} c_m k^m (ks)^{a-m}, & 0 < ks \leq 1, \\ c_m k^m (ks)^{-1/2-m}, & ks \geq 1. \end{cases}$$

Then, where $q := (2n$

(N_1,N_2) for the meshes Λ_{N,L_j,l,q_j} and $\Lambda_{N,L_j,l,q_{j+1}}$, respectively. Our approximation space $V_{N,n}$ is then the linear span of $\bigcup_{j=1,\ldots,n}\{V_{\Gamma_j^+,n}\cup V_{\Gamma_j^-,n}\}$. The total number of the degrees of freedom is $\mathcal{M}_N=(n+1)\sum_{j=1}^{n}N_j^*$, where N_j^* is the sum of the values of N_1+N_2 (the number of subintervals) for the meshes Λ_{N,L_j,l,q_j} and $\Lambda_{N,L_j,l,q_{j+1}}$. Since $-1/\log(1-1/N_1)<N_1$, for $N_1\in\mathbb{N}$, and $1<q_j<n+3/2$, we see that $N_j^*<(Nq_j+1+(Nq_j+1)\log(L_j/l)/q_j+1)+(Nq_{j+1}+1+(Nq_{j+1}+1)\log(L_j/l)/q_{j+1}+1)<(2n+3)N+4+2(N+1)\log(L_j/l)$, so that

$$\mathcal{M}_N<(n+1)n\left[(2n+3)N+4+2(N+1)\log\left(\frac{\bar{L}}{l}\right)\right]<(n+1)nN\left[2n+5+3\log\left(\frac{k\bar{L}}{2p}\right)\right], \tag{4.6}$$

where $\bar{L}:=(L_1\ldots L_n)^{1/n}$.

It follows from equations (2.11) and (4.4[redacted] Theorem 2.1 (applied with $c=p$), and Theorem 4.1 that j can be approximated very [redacted]ell by an element of the approximation space $V_{N,n}$. Precisely, these equations and theor[redacted]s imply that, if the conditions of Theorem 2.1 are satisfied, then on each interval $(\tilde{L}_{j-}$[redacted]$\tilde{L}_j)$ (corresponding to side Γ_j), there exist elements s_j^+ and s_j^- of Π_{Γ}

10.9919.1130Td[()]TJ/F9310.9091 .9091Tf7.8050Td[(),)-256(it255(Th)]TJ73.429Td[(a)]TJ/F10910.9091Tf23.2950Td[(P)]TJ/F109729701Tf6.858-3.132Td

it +log($\bar{L}$[redacted]

implywh Td [(=)]TJ/F102 5 10.9091 Tff.007 0 Td [(L)

above are irregular frequencies in the sense of §3, i.e. they are values of k for which the Dirichlet problem for the Helmholtz equation in the interior W has a non-trivial solution u_D and so the integral equation (3.4) has infinitely many solutions. (For this numerical example these irregular frequencies are $k = \sqrt{m^2 + n^2}$, for $m, n \in$ N, with the corresponding non-trivial solutions u_D given by $u_D(\mathbf{x}) = \sin m x_1 \sin n x_2$.) This lack of uniqueness at a continuous level does not appear to translate to the discrete level; the Galerkin method with our approximation space, carefully tailored to j, seems to select the right solution –
9p083 -13.5eethexeiquenby Tdmataeiquen549vex-31lutpolyg59(solethod)]TJ 0 -13.549 Td3(.piecewi)-3019en549

[6]