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Department of Mathematics and Statistics

Theory and Examples of Generalised Prime Systems

Faez Ali AL-Maamori

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Abstract

A generalised prime system $\mathcal{P}$ is a sequence of positive reals $p_1, p_2, p_3, \ldots$ satisfying $1 < p_1 \leq p_2 \leq \ldots \leq p_n \leq \ldots$ and for which $p_n \to \infty$ as $n \to \infty$. The $\{p_n\}$ called generalised primes (or Beurling primes) with the products $p_1^{a_1}.p_2^{a_2}.\ldots.p_k^{a_k}$ (where $k \in \mathbb{N}$ and $a_1, a_1, \ldots, a_k \in \mathbb{N} \cup \{0\}$) forming the generalised integers (or Beurling integers).

In this thesis we study the generalised (or Beurling) prime systems and we examine the behaviour of the generalised prime and integer counting functions $\pi_{\mathcal{P}}(x)$ and $N_{\mathcal{P}}(x)$ and their relation to each other, including the Beurling zeta function $\zeta_{\mathcal{P}}(s).$

Specifically, we study a problem discussed by Diamond (see [7]) which is to determine the best possible γ in $N_{\mathcal{P}}(x) = \rho x + O(xe^{-c(\log x)^{\gamma}}),$ for some $\rho > 0,$ given that $\pi_{\mathcal{P}}(x) = \mathrm{li}(x) + O(xe^{-(\log x)^{\theta}}),$ $\theta \in (0, 1).$ We obtain the result that :

We study the connection between the asymptotic behaviour (as $x \to \infty$) of the g-integer counting function $N_{\mathcal{P}}(x)$ (or rather of $N_{\mathcal{P}}(x) - ax$) and the size of Beurling zeta function $\zeta_{\mathcal{P}}(\sigma + it)$ with σ near 1 (as $t \to \infty$). We show in the first section how assumptions on the growth of $\zeta_{\mathcal{P}}(s)$ imply estimates on the error term of $N_{\mathcal{P}}(x)$, while in the second half we find the region where $\zeta_{\mathcal{P}}(\sigma + it) = O(t^c),$ for some $c > 0$, if we assume that we have a bound for the error term of $N_{\mathcal{P}}(x)$.

Finally we apply these results to find O and Ω results for a specific example.

Declaration

I con rm that this is my own work, and the use of all material from other sources has been properly and fully acknowledged.

Faez Ali AL-Maamori

Acknowledgements

Contents

Chapter 1

Introduction

In the late nineteenth century, Weber (see [30]) de ned $N(x)$ to be a number of function of g-primes less than or equal to x and $N_P(x)$ to be the counting function of g-integers less than or equal to x (counting multiplicities). Beurling was interested to see under which conditions on N and the multiplicative structure, a Prime Number Theorem holds.

In 1937, Beurling proved (see [6]) that if $N_P(x) = ax + O$ x

The problem is to determine the best possible (i.e. largest possible) $\;;$ given $\;:$

Furthermore, we investigate the connection between the size of the Beurling zeta function $\;_P(\;\;+ it)$ with near 1 (as $t\;\;!\;\;1$) and the error term of $N_P(x)$. As part of this investigation, if we assume that $\;_P(s)$ has polynomial growth in

In the rst section of this chapter, we generalise Balanzario's result by adapting his method to show that for any $0 < \quad < 1$ there is a continuous g-prime system for which (1) and (3) hold with $\quad = \quad$. Thus we cannot (in general) make $\quad > \quad :$

In the second half of this Chapter we use the method developed by Diamond, Montgomery, Vorhauer [11] and Zhang [31] to prove by using (the theory of) probability measures that there is a *discrete* system of Beurling primes satisfying

Chapter 2

Preliminary concepts

In this chapter we will give details of some relevant concepts and known results which we shall need in Chapters 3-6. In particular, for the definitions of generalised prime systems (especially the continuous version) we need the Riemann-Stieltjes integral and Riemann-Stieltjes convolution.

In the second half of this chapter we summarize some (relevant) results about the Riemann-Zeta function. In particular, we will give a brief survey of some of the known lower bounds for the Riemann-Zeta function in the critical strip $0 < \sigma < 1$. We consider also the upper bounds for the Riemann-Zeta function which are unconditional bounds in that strip and those which are conditional on the unproved Riemann Hypothesis.

We begin with the Riemann-Stieltjes integral.

2.1 Riemann-Stieltjes integral

Let f and α be bounded (real or complex) functions on $[a, b]$. Let $P = \{x_0, x_1, x_2, \cdots, x_n\}$ be a partition of $[a, b]$ and let $t_k \in [x_{k-1}, x_k]$ for $k = 1, 2, \cdots, n$. We define a Riemann-Stieltjes sum of f with respect to α as

$$S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k)\left(\alpha(x_k) - \alpha(x_{k-1})\right).$$

Definition 1. *A function f is Riemann Integrable with respect to α on $[a, b]$, if there exists $r \in \mathbb{R}$ having the following property: For every $\epsilon > 0$, there exists a* *partition* P *of* $[a; b]$ *such that for every partition* P *ner than* P *and for every choice of the points* t_k *in* $[x_{k\ 1}; x_k]$; *we have* bounded variation with $f(x) = 0, \forall x \in (-\infty, 1)$. Let $S^+ \subset S$ such that for any $f \in S^+$, f is an increasing function. For $a \in \mathbb{R}$, let $S_a = \{f \in S : f(1) = a\}$ and $S_a^+ = S_a \cap S^+$.

Definition 3. *For any $f, g \in S$, we define the convolution (or Riemann-Stieltjes convolution) by*

$$(f * g)(x) = \int_1^x f\left(\frac{x}{t}\right) dg(t).$$

We note that $(S, *)$ is a commutative semigroup and the identity (with respect to $*$) is $i(x) = 1$ for $x \geq 1$ and zero otherwise.

We require the following properties from the literature which are necessaryureW the liTJ/F

2.2 The Riemann zeta function

We will move our attention to the Riemann zeta function which we need for later chapters. In particular, we shall give a brief survey of some of the known results for the order of the Riemann Zeta function in the critical strip $0 < \quad < 1.$ We consider both unconditional results and those results conditional upon the Riemann hypothesis.

De nition 4. *The Riemann zeta function is de ned for* $<s > 1$

$$(s) = \sum_{n=1} \frac{1}{n^s}.$$

The above series converges absolutely and locally uniformly in the half-plane $<s > 1$ and de nes a holomorphic function here. Moreover, (s) has an analytic continuation to the whole complex plane except for a simple pole at 1 with residue 1 and is of nite order (i.e. $(\quad + it) = O(t^A);$ for some $A > 0$ dependent on). The Riemann zeta function (s) had been studied by Euler (1707-1783) as a function of real variable s. The notion of $(s$

the Euler product) nor for $\Re s < 0$ (by the Functional Equation) except for so called 'trivial zeros' at $-2n$ ($n \in \mathbb{N}$). Furthermore, it is well known that no zeros of $\zeta(s)$ lie on either of the lines $\Re s = 1$ and $\Re s = 0$ (see [29]). Note that $\zeta(s)$ is the *Mellin transform* of $[x]$ (see [2]).

Notation

We define the big oh notation O (or $\ll$), little oh notation o, asymptotic equality of functions $\sim$ and Ω notation as follows:

Definition 5. *If $g(x) > 0$ for all $x \geq a$, we write*

$$f(x) = O(g(x)) \quad \text{or} \quad f(x) \ll g(x),$$

to mean that the quotient $\frac{f(x)}{g(x)}$ is bounded for $x \geq a$; that is there exists a constant $M > 0$ such that

$$|f(x)| \leq M g(x), \quad \text{for all} \quad x \geq a.$$

An equation of the form $f(x) = h(x) + O(g(x))$ means that $f(x) - h(x) = O(g(x))$.

Definition 6. *Let $g(x) > 0$ for all $x \geq a$, then the notation*

$$f(x) = o(g(x)) \quad \text{as} \quad x \to \infty,$$

means that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.$$

An equation of the form $f(x) = h(x) + o(g(x))$ as $x \to \infty$ means that $f(x) - h(x) = o(g(x))$ as $x \to \infty$.

Definition 7. *Let $g(x) > 0$ for all $x \geq a$. If*

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1,$$

we say that $f(x)$ is asymptotic to $g(x)$ as $x \to \infty$, and write $f(x) \sim g(x)$ as $x \to \infty$.

We define Ω notation as follows:

Definition 8. *Let F, G be functions defined on some interval (a, ∞) with $G \geq 0$. We write*

$$F(t) = \Omega(G(t));$$

to mean the negation of the $F(t) = o(G(t))$. That is, there exist a constant $c > 0$ such that $|F(t)| \geq cG(t)$ for some arbitrarily large values of t.

Further, we write $F(t) = \Omega_+(G(t))$ and $F(t) = \Omega_-(G(t))$ if there exist a constant $c > 0$ such that $F(t) \geq cG(t)$ and $F(t) \leq -cG(t)$ hold respectively for some arbitrarily large values of t.

*We write $F(t$

This will be used to facilitate the proof of a result in Chapter 6 as part of our purpose in that chapter.

Proposition 2.4. *For* $\frac{3}{4} \quad\quad 1 \quad \frac{\log\log\log N}{2\log\log N}$*; we have*

$$\max_{1<t<N} j \;(\quad + it) j \quad \exp \quad (1 + o(1)) \frac{(\log N)^{1}}{16(1 \qquad) \log\log N} \quad ;$$

for $N \quad N_0$ *independent of* $:$

Proof. Take $\frac{3}{4}$

Thus,

$$\log \quad {}_{;2}(2 \quad 3 \qquad P) \qquad \sum_{p \quad P} \log(1 + \frac{1}{p}) \qquad \sum_{p \quad P} \frac{1}{p} \qquad \frac{1}{2} \sum_{p \quad P} \frac{1}{p^2} \qquad \sum_{p \quad P} \frac{1}{p} \qquad ;$$

for some absolute constant $\quad > 0$, since $x \quad \log(1 + x) \quad x \quad \frac{x^2}{2}$; for $0 \quad x \quad 1$:

We end the proof of the Proposition by showing that for every $\quad > 0$; and $\frac{3}{4} \qquad 1 \quad \frac{\log\log\log N}{2\log\log N}$; we have

$$\sum_{p \quad P} \frac{1}{p} \qquad (1 \qquad) \frac{(\log N)^1}{8(1 \qquad) \log\log N}; \quad \text{for} \quad N \qquad N_0(\ ):$$

Now, we have

$$\sum_{p \quad P} \frac{1}{p} = \int_2^P t \quad d \ (t) = \frac{(P)}{P} + \int_2^P \frac{(t)}{t^{\ +1}} dt:$$

By the Prime Number Theorem $\ (x) \quad (1 \qquad) \frac{x}{\log x}$ for $x \quad x_0(\ )$. This tells us that,

$$\sum_{p \quad P} \frac{1}{p} \qquad (1 \qquad) \frac{P^1}{\log P} + \ (1 \qquad) \int_2^P \frac{t}{\log t} dt \qquad :$$

for some absolute constant $\quad > 0$: Here

$$(1 \qquad) \int_2^P \frac{t}{\log t} dt \qquad \frac{(1 \qquad)}{\log P} \int_2^P t \quad dt = \frac{(1 \qquad)}{(1 \qquad) \log P} \ P^1 \qquad 2^1$$

Thus, for any $\quad > 0$, and $P \quad P_0(\ )$, we have

$$\sum_{p \quad P} \frac{1}{p} \qquad (1 \qquad) \frac{P^1 \qquad 2}{\log P} + \ (1 \qquad) \frac{P^1 \qquad 2}{(1 \qquad) \log P} \qquad = (1 \qquad) \frac{P^1 \qquad 2}{(1 \qquad) \log P} \qquad :$$

Now, we have $P \quad \frac{1}{8} \log N$: So,

$$\frac{P^1 \qquad 2}{(1 \qquad) \log P} \qquad \frac{(\log N)^1}{8(1 \qquad) \log\log N};$$

when $1 \qquad \frac{\log\log\log N}{2\log\log N}$ (actually, for $(1 \qquad) \log\log N \quad 1$). Therefore, from the above for $1 \qquad \frac{\log\log\log N}{2\log\log N}$; we have

$$\max_{1<t<N} j \ (\quad + it)j \qquad (N^{\frac{1}{8}}) \quad 1 \quad \max_{n \quad N^{\frac{1}{8}}} \sqrt{\qquad (n)} \quad 1$$

$$\exp \left(\frac{1}{2} \sum_{p \quad P} \frac{1}{p} \qquad \right) \quad 1 \quad \exp \quad \frac{(1 + o(1))(\log N)^1}{16(1 \qquad) \log\log N} \quad ;$$

for $N \quad N_0$ independent of $\ :$ The proof of Proposition 2.4 is completed. □

O results for (s) in the critical strip

with $B = 100$ (see [27] page 98). More research on this subject has been done to improve (2.1). In 1975, Elson proved (2.1) with $B = 86$ and $A = 2100$; see [12]. Ching in his paper (1999) improved this obtaining (2.1) with $B = 46$ and $A = 175$. Moreover, Heath Brown (in unpublished work (see page 135 in [29])) proved (2.1) with $B = 18.8$ and some $A > 0$.

O results for (s) on the Riemann Hypothesis

If we assume the truth of the unproved Riemann Hypothesis the bounds can be improved signi cantly. This will give us the strongest conditional upper bound for the Riemann Zeta function available at present in the critical strip $\frac{1}{2} \qquad 1.$ For the cases in which $= \frac{1}{}$

Chapter 3

Beurling prime systems

In this chapter we give the necessary background to Beurling (or generalised) prime systems and the associated Beurling zeta function. It is beneficial to give historical context to this subject.

In the late nineteenth century, Weber (see [30]) defined $N(x)$ to be the number of the integral ideals in a fixed algebraic number field F with the norm not exceeding x and proved that $N(x) = ax + O(x^{\theta});$ as $x \to \infty$ for some $a > 0$ and $\theta < 1.$ Early in the twentieth century, Landau (see [22]) used Weber's result and the multiplicative structure to prove the Prime Ideal Theorem, which asserts that the number of the distinct prime ideals of the ring of integers in an algebraic number field F with the norm not exceeding x is a asymptotic to $\frac{x}{\log x};$ as x tends to infinity. His result showed that the only `additive' result needed was Weber's.

3.1 Discrete g-prime systems

In 1937, Beurling (see [6]) considered number systems with only multiplicative structure, and was interested in finding conditions over the counting function of integers $N(x)$ which ensure the validity of the Prime Number Theorem. Beurling introduced generalised prime systems as follows:

Definition 9. *A generalised prime system $\mathcal{P}$ is a sequence of positive reals $p_1; p_2; p_3; \ldots$ satisfying $1 < p_1 \le p_2 \le \ldots \le p_n \le \ldots$ and for which $p_n \to \infty$*

as $n \to \infty$.

The numbers $\{p_n\}_{n \geq 1}$ are called *generalised primes* (or *Beurling primes*). The associated system of *generalised integers* (or *Beurling integers*) $N = \{n_i\}_{i \geq 1}$ can be formed from these. That is, the numbers of the form

$$p_1^{a_1}.p_2^{a_2}.\dots.p_k^{a_k} \qquad (\ )$$

where k 2:p :p

This infinite product may be formally multiplied out to give the Dirichlet series $\zeta_P(s) = \sum_{n \in N} \frac{1}{n^s}$. This is also the *Mellin transform* of N_P.

The important question in this work is: how do the distributions of P and N relate to each other?

Much of the research on this subject has been about connecting the asymptotic behaviour of the g-prime and g-integer counting functions defined in (3.1) as $x \to \infty$. Specifically, given the asymptotic behaviour of $\pi_P(x)$, what can be said about the behaviour of $N_P(x)$? On the other hand, given the asymptotic behaviour of $N_P(x)$, what can be said about the behaviour of $\pi_P(x)$? Therefore, this research concentrates on findingP7(viour)P7(viour)P7([01 Tf 12.619 -12.454 -12.4whi14nding

2. For $P = \{2, 2, 3, 3, 5, 5, 7, 7, \ldots\}$ (each prime occurs twice), with $\zeta(\cdot)$ forming $\mathcal{N}$ to be the set of integers such that each integer occurs $d(n)$ times, where $d(n)$ is the number of divisors of n. That is,

$$\mathcal{N} = \{1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 6, 6, 6, 6, 7, 7, \ldots\},$$

therefore $\pi_P(x) = 2\pi(x) = 2\sum_{p \le x} 1$ and

$$N_P(x) = \sum_{\substack{n \le x, \\ n \in \mathbb{N}}} d(n).$$

Then the behaviour of these counting functions for large x is $N_P(x) \sim x \log x$ (see [2]) and $\pi_P(x) \sim \frac{2x}{\log x}$,(by the Prime Number Theorem).

3.2 Continuous g-prime systems

The notion of g-primes as defined earlier can be generalised in such a way that we consider $\pi_P(x)$ and $N_P(x)$ as general increasing functions not necessarily step functions. Such an extension is often referred to loosely as a `continuous' g-prime system. Indeed Beurling's Prime Number Theorem is actually proven in this general setting. In the most general form, the `continuous' g-prime systems are based on the analogue of $\Pi_P(x)$ $(= \sum_{k=1}^{\infty} \frac{1}{k} \pi_P(x^{1/k}))$ and are defined as follows:

Definition 10. *Let Π_P, N_P be functions such that $\Pi_P \in S_0^+$ and $N_P \in S_1^+$ with $N_P = \exp^* \Pi_P$. Then (Π_P, N_P) is called an* outer g-prime system*.*

Note that, if $\Pi_P \in S_0^+$, then automatically $\exp^* \Pi_P \in S_1^+$. Hence any $\Pi_P \in S_0^+$ defines an outer g-prime system. On the other hand, if $N_P \in S_1^+$, then $N_P = \exp^* \Pi_P$ for some $\Pi_P \in S_0$, but Π_P need not be increasing (see section 1.3 in [15]). Here we do not (yet) have the analogue of g-primes (i.e. $\pi_P(x)$). We introduce $\pi_P(x)$ as follows:

Definition 11. *A g-prime system is an outer g-prime system for which there exist $\pi_P \in S_0^+$ such that*

$$\Pi_P(x) = \sum_{k=1}^{\infty} \frac{1}{k} \pi_P(x^{1/k}).$$

We say N_P determines a g-prime system if there exists such an increasing $_P \in S_0$. As such by *Möbius inversion*, $_P(x)$ is given by

$$_P(x) = \sum_{k=1}^{\infty} \frac{(k)}{k} \; _P(x^{1/k}); \tag{3.2}$$

provided this series converges absolutely. To show that this sum always converges for $_P \in S^+$, we let $a_k = \frac{(k)}{k}$ and let $b_k = \; _P(x^{1/k})$: The partial sums of the a_k are bounded in magnitude by q (some $q > 0$) since $\sum_{k=1}^{\infty} \frac{(k)}{k} = 0$. The sum $\sum_{k=1}^{\infty} |b_k \; b_{k+1}|$ converges since b_k decreases to zero. By Abel's summation we have

$$\sum_{k=1}^{N} a_k b_k = \sum_{k=1}^{N\,1} A_k(b_k \; b_{k+1}) + A_N b_N;$$

where $A_n = \sum_{k=1}^{n} a_k$: Therefore,

$$\sum_{k=M}^{N} a_k b_k = \sum_{k=M}^{N\,1} A_k(b_k \; b_{k+1}) + A_N b_N \quad q \sum_{k=M}^{N\,1} |b_k \; b_{k+1}| + q\,|b_N|:$$

This shows that (3.2) always converges whenever $_P$ is increasing.

In general though, $_P(x)$ (as given by (3.2)) need not be increasing (see example 2 in this section). We make the following definitions (see [4] and [15]):

Definition 12. *For an outer g-prime system* $(_P; N_P)$, *let* $_P = \; _{PL}$: *That is,*

$$_P(x) = \int_1^x \log t \; d \; _P(t);$$

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denote the generalized

P.

(

We can write this as

$$P(x) = \sum_{n=1}^{1} \sum$$

By (3.2), we find

$$\pi_P(x) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \Pi_P(x^{1/k}) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \int_1^{x^{\frac{1}{k}}} \frac{t - \frac{1}{t}}{\log t}\, dt$$

$$= \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \int_1^{x} \frac{u^{\frac{1}{k}} - u^{-\frac{1}{k}}}{\log u}\, u^{\frac{1}{k}-1}\, du = \int_1^x \frac{1}{u \log u} \sum_{k=1}^{\infty} \frac{\mu(k)}{k} (u^{\frac{2}{k}} - 1)\, du$$

$$= \int_1^x \frac{1}{u \log u} \left(\sum_{m=1}^{\infty} \frac{2^m (\log u)^m}{m!} \sum_{k=1}^{\infty} \frac{\mu(k)}{k^{1+m}} \right) du$$

$$= \int_1^x \frac{1}{u} \sum_{m=1}^{\infty} \frac{2^m (\log u)^{m-1}}{m!\, \zeta(1+m)} du; \quad \text{for } x \geq 1.$$

This shows that $\pi_P \in S_0^+$ and therefore we have a g-prime system. Moreover, in this case we have $N_P(x) = x^2$; since by (3.3) we have

$$\int_1^x \log t\, dN_P(t) = \int_1^x N_P\left(\frac{x}{t}\right) d\psi_P(t) = \int_1^x N_P\left(\frac{x}{t}\right)\left(t - \frac{1}{t}\right) dt$$

$$= x \int_1^x N_P(u) \left(\frac{x}{u} - \frac{u}{x}\right) \frac{du}{u^2}.$$

That is,

$$N_P(x) \log x - \int_1^x \frac{N_P(t)}{t} dt = x^2 \int_1^x \frac{N_P(u)}{u^3} du - \int_1^x \frac{N_P(u)}{u} du.$$

By differentiating and simplifying, we get $\frac{d}{dx}\left(\frac{N_P(x)\log x}{x^2}\right) = \frac{N_P(x)}{x^3}$. Therefore, $\log N_P(x) = 2\log x + c$; but $N_P(1) = 1$; which means $c = 0$.

2. Let $\Pi_P(x) = \int_1^x \frac{1 - t^{-c}}{\log t}\, dt$; $x \geq 1$; and $c > 0$. This means that Π_P and π_P are increasing, where

Working with $\psi_P(x)$ is often more convenient than working with $\pi_P(x)$. One reason is due to the following direct link between ζ_P and ψ_P

$$-\frac{\zeta_P'}{\zeta_P}(s) = \int_1^\infty x^{-s}\, d\psi_P(x).$$

From Definition 12 above, the following statements

$$\pi_P(x) = \mathrm{li}(x) + O(x^{\theta+\epsilon}), \ \forall \epsilon > 0 \tag{3.4}$$

and

$$\psi_P(x) = x + O(x^{\theta+\epsilon}), \ \forall \epsilon > 0, \tag{3.5}$$

are equivalent for $\theta \in [0, 1)$. Furthermore, we see that $\Pi_P(x) \geq \pi_P(x)$ and

$$\begin{aligned}
\Pi_P(x) - \Pi_P(\sqrt{x}) &= \sum_{k=1}^{\infty} \frac{\pi_P(x^{\frac{1}{k}})}{k} - 2\sum_{k=1}^{\infty} \frac{\pi_P(x^{\frac{1}{2k}})}{2k} \\
&= \sum_{k \geq 1} \frac{\pi_P(x^{\frac{1}{k}})}{k} - 2\sum_{k\ \mathrm{even}} \frac{\pi_P(x^{\frac{1}{k}})}{k} \\
&= \sum_{k \geq 1} \frac{(-1)^{k-1}\pi_P(x^{\frac{1}{k}})}{k} \leq \pi_P(x),
\end{aligned}$$

since π_P is increasing. This tells us that

$$0 \leq \Pi_P(x) - \pi_P(x) \leq \Pi_P(\sqrt{x}).$$

Thus, $\Pi_P(x) = \pi_P(x) + O(\Pi_P(\sqrt{x}))$. Then the following statements

$\Pi_P(x) = \mathrm{li}(x) + O(x^{\theta+\epsilon}), \ \forall \epsilon > 0$ and $\psi_P(x) = x +$

1. In 1937, Beurling (see [6]) proved that

$$N_P(x) = ax + O\left(\frac{x}{(\log x)^{\gamma}}\right) \text{ for some } \gamma > \frac{3}{2} \Rightarrow \pi_P(x) \sim \frac{x}{\log x},$$

(generalises Prime Number Theorem), and he showed by example that the result can fail for $\gamma = \frac{3}{2}.$

2. In 1977, Diamond (see [10], Theorem 2) as a type of converse of Beurling's PNT, showed the following: *suppose that* $\int_2^{\infty} t^{-2} \left|\pi_P(t) - \frac{t}{\log t}\right| dt < \infty.$ *Then there exists a positive constant c such that* $N_P(x) \sim cx$ *as* $x \to \infty.$ Diamond in his work was seeking weakest possible conditions on $\pi_P(x)$ which are sufficient to deduce that $N_P(x) \sim cx$ as $x \to \infty.$ So, for example it follows from Diamond's work that

$$\pi_P(x) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^{1+\delta}}\right) \text{ for some } \delta > 0 \Rightarrow N_P(x) \sim cx.$$

3. In 1903, Landau (see [22]) proved that

$$N_P(x) = ax + O(x^{\theta}), (\theta < 1), \tag{3.6}$$

implies $\pi_P(x) \sim \frac{x}{\log x}.$ Furthermore, he proved that (3.6) implies

$$\pi_P(x) = \mathrm{li}(x) + O(xe^{-k\sqrt{\log x}})$$

for some $k > 0.$

4. In 2006, Diamond, Montgomery and Vorhauer (see [11]) showed Landau's result is best possible.That is, they proved that there is a discrete g-prime system for which (3.6) holds but

$$\pi_P(x) = \mathrm{li}(x) + \Omega(xe^{-q\sqrt{\log x}}) \text{ for some } q > 0.$$

5. In 1969, Malliavin (see [24]) showed that for $\alpha \in (0,1)$ and $a, c > 0$

$$N_P(x) = ax + O(xe^{-c(\log x)^{\alpha}})$$

$x) = x \quad O \quad k(\log$

6. In his paper 1970, Diamond (see [7]) improved Malliavin's result and conversely he showed that if

$$_P(x) = \mathrm{li}(x) + O(xe^{\,c(\log x)});$$

holds for 2 (0

then by Diamond's result we see ( ) $\frac{}{1+}$: Further, from Balanzario's result we see that $(\frac{1}{2})$ $\frac{1}{2}$: Diamond and Bateman [5] raised the interesting problem to determine ( ) for 0 < < 1:

In our work we study Balanzario's method in his paper and modify it to show (by adapting the method) that there is a (continuous) g-prime system for which (3.7) and (3.8) hold with = (any 0 < < 1), this showing ( ) : Further-more, we prove that there is a *discrete g-prime system* with the same property ( ) : This is more challenging since we need $_P(x)$ de ned as a step function. For this we use the method developed by Diamond, Montgomery, Vorhauer [11] and Zhang [31] to prove by using (the theory of) probability measures that there is a discrete system of Beurling primes satisfying this same property. We illustrate this in Chapter 4.

From the known results (listed above), we see that for 0 ; < 1 the statement

$$_P(x) = x + O(x\ ); \tag{3.9}$$

does not necessarily imply

$$N_P(x) = \ x + O(x\ ); \quad > 0. \tag{3.10}$$

Actually, the example given in chapter 6 shows that (3.9) $\Rightarrow$ (3.10) is false for g-prime systems. For general g-prime systems that (3.10) does not imply (3.9) for discrete g-prime systems follows from a result of Diamond, Montgomery, Vorhauer paper [11] shows by using the probabilistic construction that there is a discrete system for which (3.10) does not imply (3.9).

Discrete g-prime systems where the functions $N_P(x)$ and $_P(x)$ are simulta-neously `well-behaved', that is (3.9) and (3.10) hold have been investigated by Hilberdink (see [17]). In particular, if (3.9) and (3.10) hold then one of or is at least $\frac{1}{2}$ (see Theorem 1: in [16]). We shall require the following two results in our subsequent work.

Lemma 3.2. *Suppose that for some $\in [0, 1)$; we have* (3.9) *holds. Then $_P(s)$ has analytic continuation to the half-plane $H = \{s \in \mathbb{C} : \Re s > \}$ except for a simple (non removable) pole at $s = 1$ and $_P(s) \neq 0$ in this region.*

Proof. See first part of Theorem 2.1 in [17]. □

Lemma 3.3. *Suppose for $0 \quad ; \quad < 1$ both* (3.9) *and* (3.10) *hold. Then for $> = \max\{ \; ; \; \}$; and uniformly for $+$ (any > 0), $_P(s)$ is of zero order for $>$. Furthermore,*

$$\frac{'_P}{_P}(s) = O\left((\log t)^{\frac{1}{1-}+}\right);$$

and

$$_P(s) = O\left(\exp\{(\log t)^{\frac{1}{1-}+}\}\right);$$

for all > 0.

Proof. The proof of this lemma is given for discrete g-prime systems [17, Theorem 2.3], but holds more generally for outer g-prime systems as well (since no use is made of $_P(x)$). □

Assume that we have a discrete g-prime system such that (3.10) holds with $< \frac{1}{2}$. It was shown in [16] that this implies $_P(s)$ has non-zero order for $< < \frac{1}{2}$. This shows that there is a link between the asymptotic behaviour (as $x \to \infty$) of the g-integer counting function $N_P(x)$ and the size of Beurling zeta function $_P($

Chapter 4

Examples of continuous and discrete g-prime systems

In this chapter we introduce a problem discussed by Diamond [7](as mentioned brie y in section 3.3), which is the following:

Assume $_P(x)$ li(x) $xe^{(\log x)}$; for some $2\ (0;1)$; so that

$$N_P(x) = x + O(xe^{c(\log x)}); \tag{4.1}$$

for some $;c > 0$ and > 0: The problem is to determine the best possible ; given : So, let ( ) be the supremum of such over all systems satisfying (3.7) for given $2\ (0;1)$: It follows from Malliavin's result that ( ) 10 : Diamond in 1970 (see [8]) proved that ( ) $\frac{}{1+}$: In 1998, Balanzario [3] proved (by giving a concrete continuous example) that there exists a continuous g-prime system for which $= = \frac{1}{2}$ in (3.7) and (3.8). Thus, $(\frac{1}{2})$ $\frac{1}{2}$:

In the rst section of this chapter, we generalise Balanzario's result by adapting his method to show that for any $0 < < 1$ there is a continuous g-prime system for which (3.7) and (3.8) hold with $=$: Thus, ( ) .

In the second section we do more challenging work using the theory developed by Diamond, Montgomery, Vorhauer [11] and Zhang [31] to prove by using (the theory of) probability measures that there is a discrete g-prime system for which (3.7) and (3.8) hold with $=$: Thus, ( ) for discrete g-prime systems.

4.1 Continuous g-prime System

Theorem 4.1. *Let* $0 < \quad < 1$*. Then there exists an outer g-prime system* P *for which*

$$_P(x) = \mathrm{li}(x) + O(xe^{\quad(\log x)}); \tag{4.2}$$

and

$$N_P(x) = \quad x + \quad (xe^{\quad c(\log x)}); \tag{4.3}$$

for some positive constants and c*. Thus,* $(\ )$ $:$

We de ne $_P(x)$ (of g-primes) as in Balanzario's paper by

$$_P(x) = \int_1^x \frac{1 \quad t^{\quad k}}{\log t} \ (t)dt; \tag{4.4}$$

where

$$(t) = 1 \quad \sum_{n>n_0} {}_n \frac{\cos(b_n \log t)}{t^{a_n}}; \quad t \quad 1:$$

Here k and n_0 are positive constants and $_n; a_n$ and b_n are sequences to be chosen. In fact, we shall take $k = 4; n_0 = 3;$ $_n = \frac{2}{n^2};$ but it is notationally more convenient to use $k; n_0$ and $_n$. The sequences b_n and a_n are de ned (in terms of another sequence x_n) as follows:

$$b_n = \exp f(\log x_n) \ g \quad \text{and} \quad a_n = \frac{1}{(\log x_n)^{1}} \quad = \frac{1}{(\log b_n)} \quad ;$$

where $= \frac{1}{\ }$ 1. Here $x_n = \exp f e^{a!^n} g;$ for some $a > 0$ and $! > 1$ which we shall choose later. Note that, a_n $!$ 0 while b_n $!$ 1 as n $!$ $1:$ So, $x_{n+1} = \exp f(\log x_n)^{!} g;$ with $x_1 = \exp f e^{a!} g:$ We choose $!$ so that $!$ $1:$

The function $_P(x)$ is increasing since for t $1;$

$$\sum_{n>n_0} {}_n \frac{\cos(b_n \log t)}{t^{a_n}} \quad \sum_{n>n_0} \frac{2}{n^2} \quad 1:$$

First, we show that (4.2) holds.

Proposition 4.2. *If* $_P(x)$ *is given by* (4.4)*, then*

$$_P(x) = \mathrm{li}(x) + O(xe^{\quad(\log x)}):$$

Proof. We have

$$_P(x) = \int_1^e \frac{1 \quad t^{\quad k}}{\log t} \quad (t)dt + \int_e^x \frac{1 \quad t^{\quad k}}{\log t} \quad (t)dt;$$

the rst integral is just $O(1)$, therefore we get

$$_P(x) = \int_e^x \frac{1 \quad t^{\quad k}}{\log t}dt \quad \sum_{n>n_0} \quad _n \int_e^x \frac{1 \quad t^{\quad k}}{\log t} \quad \frac{\cos(b_n \log t)}{t^{a_n}}dt + O(1)$$

$$= \int_e^x \frac{dt}{\log t} \quad \sum_{n>n_0} \quad _n \int_e^x \frac{\cos(b_n \log t)}{t^{a_n} \log t}dt + O(1);$$

because $k > 1$. Now we show that the second term is $O(xe^{\quad (\log x)}\quad):$ Notice that

$$\int_e^x \frac{\cos(b_n \log t)}{t^{a_n} \log t}dt = \int_1^{\log x} \frac{\cos(b_n t)}{t} e^{t(1 \quad a_n)}dt$$

$$= \quad \frac{\sin(b_n t)}{tb_n} e^{t(1 \quad a_n)} \Big|_1^{\log x} \quad \frac{1}{b_n} \int_1^{\log x} \frac{\sin(b_n t)}{t} e^{t(1 \quad a_n)}(1 \quad a_n \quad \frac{1}{t})dt$$

$$2\frac{x^{1 \quad a_n}}{b_n \log x}$$

at all points of continuity of $N_P(x)$. The main difficulty will be to show (4.3), that is to find the result for N_P. The proof forms the rest of this section.

Now, let $M_P(x) = \int_1^x N_P(t)dt$. Then for $x > 1$

$$M_P(x) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \zeta_P(s)\frac{x^{s+1}}{s(s+1)}ds; \quad b > 1.$$

We already know that (4.1) holds for some $\frac{}{1+}$ (see result 5 in 3.3). So, to prove that equation (4.3) is true it suffices to show that for some positive constants $c;$

$$M_P(x) = \frac{}{2}x^2 + O\left(x^2 e^{-c(\log x)}\right). \tag{4.5}$$

Actually, if (4.3) does not hold then

$$N_P(x) = x + o(xe^{-c(\log x)});$$

so that,

$$M_P(x) = \int_1^x \{t + o(te^{-c(\log t)})\}dt = \frac{}{2}x^2 + o\left(x^2 e^{-c(\log x)}\right);$$

which contradicts (4.5). So, (4.3) must hold if (4.5) holds. Our aim is therefore to prove that (4.5) is true for some $c; > 0$. For this purpose we estimate the integral of $M_P(x)$ and the simplest way to do so is by calculating the contribution of the singularities of the integrand $g(s) = \zeta_P(s)\frac{x^{s+1}}{s(s+1)}$. We rewrite $\zeta_P(s)$ as an infinite product to enable us to read off the singularities of $g(s)$. The sequences $\{a_n\}$ and $\{b_n\}$ are defined earlier will give us the position of the singularities of $\zeta_P(s)$ in the complex plane, and from this we can deduce the statement (4.5). Extend the sequences $a_n; b_n$ and $_n$ by defining for $n > n_0$, $a_{-n} = a_n$, $b_{-n} = -b_n$ and $_{-n} = _n$. Then we use the following proposition to rewrite the zeta function as required.

Proposition 4.3. *For* $\Re(s) > 1;$

$$\zeta_P(s) = \frac{s+k-1}{s-1}\prod_{|n|>n_0}\left(1 - \frac{k}{s-1+a_n-ib_n+k}\right)^{-\frac{n}{2}}. \tag{4.6}$$

Remark: Recall the definition of $\phi(t)$ and let

$$\phi_N(t) = 1 - \sum_{n_0 < n \le N} \mu_n \frac{\cos(b_n \log t)}{t^{a_n}}; \quad t \ge 1.$$

Then $\phi_N(t)$ converges uniformly to $\phi(t)$ for $t \ge 1$ since $|\phi(t) - \phi_N(t)| \le \sum_{n>N} \frac{2}{n^2} \le \frac{2}{N}$.

Proof of Proposition 4.3. We have

$$\frac{\cos(b \log t)}{t^a} \frac{1 - t^{-k}}{\log t} = \frac{1}{2}(t^{-a+ib} + t^{-a-ib})\frac{1 - t^{-k}}{\log t}.$$

So, for $\Re(s) > 1$, we have

$$\frac{d}{ds}\int_1^\infty t^{-s}\frac{\cos(b \log t)}{t^a} \frac{1 - t^{-k}}{\log t} dt$$
$$= -\frac{1}{2}\int_1^\infty (t^{-s-a-ib} + t^{-s-a+ib} - t^{-s-a-ib-k} - t^{-s-a+ib-k})dt$$
$$= -\frac{1}{2}\left[\frac{1}{s-1+a+ib} - \frac{1}{s-1+a+ib+k} + \frac{1}{s-1+a-ib} - \frac{1}{s-1+a-ib+k}\right]$$
$$= -\frac{1}{2}\left(\frac{d}{ds}\log\frac{s-1+a+ib}{s-1+a+ib+k} + \frac{d}{ds}\log\frac{s-1+a-ib}{s-1+a-ib+k}\right)$$
$$= -\frac{d}{ds}\log\left\{\left(1 - \frac{k}{s-1+a+ib+k}\right)^{\frac{1}{2}}\left(1 - \frac{k}{s-1+a-ib+k}\right)^{\frac{1}{2}}\right\}.$$

Hence, we have

$$\int_1^\infty t^{-s}\frac{\cos(b \log t)}{t^a} \frac{1 - t^{-k}}{\log t} dt$$
$$= -\log\left\{\left(1 - \frac{k}{s-1+a+ib+k}\right)^{\frac{1}{2}}\left(1 - \frac{k}{s-1+a-ib+k}\right)^{\frac{1}{2}}\right\} + \textit{constant}.$$

By taking the limit as $\Re(s)$ tends to infinity we see that the constant of integration is zero. Taking $a = b = 0$ gives

$$\int_1^\infty t^{-s}\frac{1 - t^{-k}}{\log t} dt = \log\frac{s+k-1}{s-1}.$$

Thus from the definition of $\phi_N(t)$, we get

$$\int_1^\infty t^{-s}\frac{1 - t^{-k}}{\log t}\phi_N(t)dt = \log\frac{s+k-1}{s-1} + \sum_{n_0 < |n| \le N} \mu_n \log\left(1 - \frac{k}{s-1+a_n-ib_n+k}\right)^{\frac{1}{2}}$$

$$= \log \left\{ \frac{s+k-1}{s-1} \prod_{n_0<|n|<N} \left(1 - \frac{k}{s-1+a_n-ib_n+k}\right)^{-\frac{\alpha_n}{2}} \right\}.$$

By taking the limit as $N \to \infty$, we conclude the proof since $\Pi_N(t) \to \Pi(t)$ as $N \to \infty$ and $\log \zeta_P(s) = \int_1^\infty t^{-s}\, d\Pi_P(t)$. □

The representation of $\zeta_P(s)$ given by (4.6) holds not only in the half plane $\Re(s) > 1$; but also in a larger region. Let D be the region defined by

$$D = \{s = \sigma + it \in \mathbb{C} : \sigma > -k+2; s \neq \theta(1-a_n+ib_n) + (1-\theta)(1-a_n+ib_n-k);$$

$$\text{for any } 0 \leq \theta \leq 1; |n| \leq n_0\}.$$

By a theorem of Weierstrass on the uniform convergence of analytic functions, the function

$$\varphi(s) = \prod_{|n|>n_0} \left(1 - \frac{k}{s-1+a_n-ib_n+k}\right)^{-\frac{\alpha_n}{2}};$$

is analytic in D. The equation

$$\zeta_P(s) = \frac{s+k-1}{s-1}\varphi(s); \quad \sigma > 1;$$

gives us an analytic continuation of $\zeta_P(s)$ to D with $s = 1$ removed, where $\zeta_P(s)$ has a simple pole. Notice that, since the zeros of $\varphi(s)$ are of fractional order, we avoid problems of multiple-valuedness by restricting the domain of definition of $\zeta_P(s)$ to D. We try to give a suitable upper bound for $|\zeta_P(s)|$ in the extended domain of definition. For this purpose we need the following

Proposition 4.4. *If $s = \sigma + it$ is such that $\sigma > -k+2$; $\alpha = \sum_{n>n_0} \alpha_n$; and $s \in D$; then*

$$|\varphi(s)| \leq (k+1)e^{\alpha}.$$

Proof. For $s = \sigma + it$, we find an upper bound for $\varphi(s)$ which holds for arbitrary positive sequences $\{a_n\}$ and $\{b_n\}$ such that $\{a_n\}$ is decreasing to zero and $\{b_n\}$ is increasing to ∞. We have

$$\begin{aligned} b_{n+1} - b_n &= \exp\{(\log x_{n+1})^\gamma\} - \exp\{(\log x_n)^\gamma\} \\ &= \exp\{(\log x_n)^{\prime}\gamma\} - \exp\{(\log x_n)^\gamma\} \geq \delta; \ (\text{some } \delta > 0); \end{aligned}$$

where δ depends on λ and ω. So, we choose ω sufficiently large such that $\delta \geq 2k$. Therefore the interval $(t-2k, t+2k)$ contains at most one element of $\{b_n\}$. We call this element (if exists) by $b_n(t)$, so we can write

$$|\varphi(s)| = \left|1-\frac{k}{s-1+a_{n(t)}-ib_{n(t)}+k}\right|^{-\frac{\lambda_{n(t)}}{2}} \prod_{|n|>n_0;\, n\neq n(t)} \left|1-\frac{k}{s-1+a_n-ib_n+k}\right|^{-\frac{\lambda_n}{2}}.$$

Since $a_n > 0$, we have $|s-1+a_n-ib_n+k| \geq 1+k > 1$ and hence

$$\left|1-\frac{k}{s-1+a_{n(t)}-ib_{n(t)}+k}\right|^{-\frac{\lambda_{n(t)}}{2}} \leq \left(1+\frac{k}{1+k}\right)^{\frac{\lambda_{n(t)}}{2}} \leq 1+k.$$

Now, when $n \neq n(t)$,

$$\left|1-\frac{k}{s-1+a_n-ib_n+k}\right|^{-\frac{\lambda_n}{2}} = \exp\left(-\frac{\lambda_n}{2}\log\left|1-\frac{k}{s-1+a_n-ib_n+k}\right|\right)$$

$$= \exp\left(-\frac{\lambda_n}{2}\Re\log\left(1-\frac{k}{s-1+a_n-ib_n+k}\right)\right)$$

$$= \exp\left(-\frac{\lambda_n}{2}\Re\left(-z-\frac{z^2}{2}-\frac{z^3}{3}-\cdots\right)\right);$$

where

$$|z| = \left|\frac{k}{s-1+a_n-ib_n+k}\right| \leq \frac{k}{|\Im(s)-b_n|} = \frac{k}{|t-b_n|} \leq \frac{k}{\delta} \leq \frac{k}{2k} = \frac{1}{2}.$$

Therefore

$$|\varphi(s)| \leq (k+1)\prod_{|n|>n_0;\, n\neq n(t)} \exp\left(\frac{\lambda_n}{2}\left(|z|+\frac{|z|^2}{2}+\frac{|z|^3}{3}+\cdots\right)\right)$$

$$\leq (k+1)\exp\left\{\frac{1}{4}\sum_{|n|>n_0}\lambda_n\left(1+\frac{1}{2}+\frac{1}{4}+\cdots\right)\right\} \leq (k+1)e^{\lambda};$$

as required. □

For $k = 4$, $n_0 = 3$ and $\lambda_n = 2n^{-2}$ we have

$$|\varphi(s)| \leq 5\exp\left\{\sum_{n>3}\frac{2}{n^2}\right\} < 9, \quad \text{if } \delta > 2.$$

Corollary 4.5. *For $s \in D$ such that*

Proof.

$$j \;{}_P(s)j = \frac{s + k \quad 1}{s \quad 1}\, '(s) \qquad 9 \;\; 1 + \frac{k}{s}$$

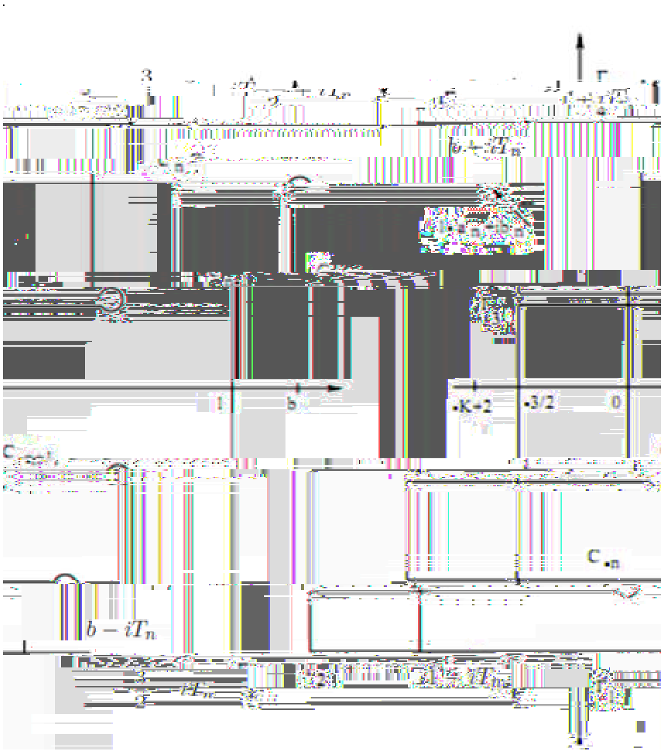

Figure 4.1:

Now we write

$$M_P(x) = I_1 + \cdots + I_5 + \sum_{n_0<|m|\le n} J_m + \{k\varphi(1)\frac{x^2}{2} + x(1-k)\varphi(0)\};$$

where

$$I_m = \frac{1}{2\pi i}\int_{\gamma_m} \Phi_P(s)\frac{x^{s+1}}{s(s+1)}ds; \quad m = 1; 2; \ldots; 5;$$

$$J_m = \frac{1}{2\pi i}\int_{C_m} \Phi_P(s)\frac{x^{s+1}}{s(s+1)}ds; \quad n_0 < |m| \le n:$$

Here, as above, C_m is the mth horizontal loop with imaginary part equal to b_m. Consider first the integral I_3. In fact, we do not have one integral but many of them. This is because the vertical segment γ_3 is broken at each horizontal loop C_m: However, on each vertical component of γ_3 the integrand is bounded by the same constant which is 45. Thus, since $R(s) = -\frac{3}{2}$ on γ_3; we have

$$|I_3| \le \frac{1}{2\pi}\int_{-\infty}^{\infty} 45\frac{x^{-\frac{3}{2}+1}}{|(-\frac{3}{2}+it)(-\frac{1}{2}+it)|}dt = O(\frac{1}{\sqrt{x}}): \tag{4.8}$$

Let $b = 1 + \frac{1}{\log x_n}$: Then $|I_2|$ and $|I_4|$ are both $O\ (\frac{x}{T_n})^2$; since

$$|I_2|;|I_4| \quad \frac{1}{2}\int_{\frac{3}{2}}^{1+(\log x_n)\ \ 1} 45\frac{x^{2+(\log x_n)\ \ 1}}{T_n^2}d \quad \frac{8}{T_n^2}x^{2+(\log x_n)\ \ 1} = O(\frac{x^2}{T_n^2}): \qquad (4.9)$$

Now we consider the integrals I_1 and I_5: Each of $|I_1|$ and $|I_5|$ is at most

$$\frac{1}{2}\int_{T_n}^{1} 45\frac{x^{2+(\log x_n)\ \ 1}}{t^2}dt \quad 8x^2\exp\left(1+\frac{1}{\log x_n}\right)^2 \frac{1}{T_n} = O(\frac{x^2}{T_n}): \qquad (4.10)$$

Therefore, we get

$$M_P(x) = \frac{x^2}{2} + \sum_{n_0<|m|\ \ n\ \ 1} J_m + \{J_{\ n} + J_n\} + O\ \frac{x^2}{T_n}\ : \qquad (4.11)$$

We estimate each term in the right hand side of (4.11) separately. Since $\log x = \log x_n + o(1)$ we get

$$\frac{x^2}{T_n} = x^2\exp\{(\log x_n)\ \} = x^2\exp\{(\log x)\ \ + o(1)\}:$$

From this and from equation (4.11) we get

$$M_P(x) = \frac{x^2}{2} + \sum_{n_0<|m|\ \ n} \ x_n + o$$

$$2\exp\left\{ (\log x)^{1 \quad \frac{1}{!}} \right\} :$$

Hence,

$$\sum_{n_0 < jmj \quad n \quad 1} J_m \quad 30\, x^2 e^{\ (\log x)^{1 \quad \frac{1}{!}}} \quad \sum_{jmj > n_0} \frac{1}{b_m^2} = O \quad x^2 e^{\ (\log x)^{1 \quad \frac{1}{!}}} \quad :$$

$\square$

We see that $1 \quad \frac{1}{!} \quad \frac{2}{+1} = \quad$ since $!$ is taken su ciently large, so equation (4.12) becomes

$$M_P(x) = \frac{x^2}{2} + f J \quad_n + J_n g + O \quad x^2 e^{\ (\log x)} \quad ; \tag{4.13}$$

It remains to study the expression $J \quad_n + J_n$. Denote by J_n^0 and J_n^{00} the integrals along the line segments C_n^0 C_n^{00} lying respectively above and below the branch cut C_n so that $J_n = J_n^0 + J_n^{00}$. Now, if we write

$$s = 1 \quad a_n + ib_n + te$$

To deal with the integral over $(0;(\log x)\quad)$ rewrite the integrand as follows:

$$\frac{{}_{P}(s)}{s(s+1)} = (s \quad 1 + a_n \quad ib_n)^{\frac{n}{2}} f_n(s); \quad \text{say};$$

where

$$f_n(s) = \frac{(s+3)}{s(s+1)(s \quad 1)(s+a_n \quad ib_n+3)^{\frac{n}{2}}} \prod_{jmj>n_0;m\neq n} 1 \quad \frac{4}{s+a_m \quad ib_m+3}^{\frac{m}{2}}: \tag{4.15}$$

Here $f_n(s)$ is analytic theo(in)28(+)]TJ/F23 11.9552 Tf 211.49 0 Td [zf

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$_2\ f_n)$m $a_n + ib_n$)

v11

$_2\ f_n$

1 $a_n + ib_n$

$(\log x)^{n}_{n\ n}$
$\log(\log X X)$

Since $J_n = J'_n + J''_n$; becomes

$$J_n = \frac{\sin\frac{\pi n}{2}\, x^{2-a_n+ib_n}}{(\log x)^{-\frac{n}{2}+1}} S_n + O\left(x^2 e^{-(\log x)^{1}}\right): \tag{4.18}$$

Since $J_{-n} = \bar{J}_n$; we have

$$J_n + J_{-n} = (J_n + \bar{J}_n) = 2\Re(J_n):$$

Our next step is to estimate the integral S_n appearing in (4.18). For this we obtain lower and upper bounds for $f_n(s)$ in $D(z_n;1)$ (that is $|s - z_n| \le 1$, $s = 1 - a_n + ib_n - y$). For the upper bound we notice that $|s| \ge b_n$: Thus

$$\left|\frac{(s+3)}{s(s+1)(s-1)}\right| \le \frac{b_n+6}{(b_n-2)^3} \le \frac{2b_n}{(b_n/2)^3} = \frac{16}{b_n^2}:$$

Also

$$|s + a_n - ib_n + 3|^{-\frac{n}{2}} > (4 - |s - 1 + a_n - ib_n|)^{-\frac{n}{2}} \ge 3^{-\frac{n}{2}} \ge 1:$$

Now we want to estimate from above the product appearing in the definition of f_n in (4.15). As in the proof of Proposition 4.4 we have

$$\left|1 - \frac{4}{s + a_m - ib_m + 3}\right| \le 1 + \frac{4}{|\Im(s) - b_m|} \le 1 + \frac{k}{\ } \le 1 + \frac{k}{2k} = \frac{3}{2}; \quad \text{for } m \ne n:$$

Thus the product in (4.15) is in modulus less than

$$\prod_{|m|>n_0; m\ne n} \left(\frac{3}{2}\right)^{\frac{m}{2}} \le \prod_{|m|>n_0} \left(\frac{3}{2}\right)^{\frac{1}{m^2}} \le \left(\frac{3}{2}\right)^{2\sum_{j=1}^{\infty}\frac{1}{j^2}} < 4:$$

Thus we have proved

Proposition 4.7. *For $|s - (1 - a_n + ib_n)| \le 1$; then $|f_n(s)| \le \frac{64}{b_n^2}$:*

This and Cauchy's inequalities give the following

Corollary 4.8. *For all $j = 1;2;3;\ldots$ $|a_{n;j}| \le \frac{64}{b_n^2}$:*

Now we estimate the lower bound for $f_n(s)$ in $D(z_n;1)$:

$$|s| \le |s - 1 + a_n - ib_n| + |1 - a_n + ib_n| \le 1 + 1 + |a_n| + |b_n| \le 3 + b_n:$$

Thus

$$\left|\frac{(s+3)}{s(s+1)(s-1)}\right| \geq \frac{|s+3|}{(b_n+4)^3} \geq \frac{|s|-3}{(b_n+4)^3} \geq \frac{b_n-1-3}{(b_n+4)^3} \geq \frac{\frac{1}{2}b_n}{(2b_n)^3} = \frac{1}{16b_n^2}.$$

Each term in the infinite product in (4.15) is

$$\left|1-\frac{4}{s+a_m-ib_m+3}\right| \geq 1-\frac{4}{|s+a_m-ib_m+3|} \geq 1-\frac{4}{|\Im(s)-b_m|} \geq 1-\frac{k}{\,} \geq 1-\frac{k}{2k} = \frac{1}{2}.$$

Therefore

$$\prod_{|m|>n_0;\, m\neq n} \left|1-\frac{4}{s+a_m-ib_m+3}\right|^{-\frac{m}{2}} \geq \prod_{|m|>n_0} \left(\frac{1}{2}\right)^{\frac{1}{m^2}} > \frac{1}{10}.$$

Thus we have

Proposition 4.9. *For $|s-(1-a_n+ib_n)| \leq 1$, we have*

$$|f_n(s)| \geq \frac{1}{160b_n^2}.$$

With all these inequalities we can estimate the integral S_n, the function occurring in (4.18), as follows:

$$\begin{aligned} S_n &= \int_0^{(\log x)^{-1}} e^{-t}t^{-\frac{n}{2}} f_n\left(1-an+ib_n-\frac{t}{\log x}\right)dt \\ &= \int_0^{(\log x)^{-1}} e^{-t}t^{-\frac{n}{2}} \sum_{j=0}^{\infty} a_{n;j}\left(-\frac{t}{\log x}\right)^j dt \\ &= a_{n;0}\int_0^{(\log x)^{-1}} e^{-t}t^{-\frac{n}{2}}dt + \sum_{j=1}^{\infty} a_{n;j}\int_0^{(\log x)^{-1}} e^{-t}t^{-\frac{n}{2}}\left(-\frac{t}{\log x}\right)^j dt. \end{aligned}$$

For the second term we get, by Corollary 4.8,

$$\begin{aligned} \left|\sum_{j=1}^{\infty} a_{n;j}\int_0^{(\log x)^{-1}} e^{-t}t^{-\frac{n}{2}}\left(-\frac{t}{\log x}\right)^j dt\right| &\leq \sum_{j=1}^{\infty} \frac{64}{b_n^2}\left(\frac{1}{\log x}\right)^j \int_0^{\infty} e^{-t}t^{-\frac{n}{2}}dt \\ &\leq \frac{1}{\log x}\, 64 \end{aligned}$$

The integral S_n in (4.18) is

$$S_n = a_{n,0} \quad (\frac{1}{2} \quad _n + 1) + O \quad \log x e^{\quad (\log x)^{1}} \qquad + O \quad \frac{1}{b_n^2 (\log x)} \quad . \qquad (4.19)$$

Since $a_{n,0} = f_n(1 \quad a_n + i b_n)$ and $(\frac{1}{2} \quad _n + 1) \to 1$ as $n \to \infty$, from Proposition 4.9 we find

$$|S_n| \quad \frac{d_0}{b_n^2} \quad 1 \quad 2 \log x e^{\quad (\log x)^{1}} \qquad \frac{d_1}{(\log x)} \qquad d e^{\quad 2(\log x)} \ , \quad d > 0$$

for some $d_0, d_1 > 0$ and for x sufficiently large, that is for n is sufficiently large (since x is a sequence depending on n). We use this lower bound of the integral S_n appearing in equation (4.18). Now consider the other factor in that equation,

$$\frac{\sin \frac{\quad n}{2}}{} x^{2 \quad a_n} \quad \frac{1}{\log x}^{\frac{\quad n}{2} + 1} \qquad \frac{\quad n}{} x^2 e^{\quad a_n \log x} \quad \frac{1}{2(\log x)^2}$$

$$a x^2 e^{\frac{\log x}{(\log x_n)^{1}}} \quad \frac{1}{(\log x)^2 (\log \log x_n)^2}; \quad \text{for some} \quad a > 0,$$

using $_n = \frac{1}{n^2}$ and $n \quad \frac{\log \log x_n}{\to \quad 1}$. From the above bound on S_n and (4.18), we get

$$|J_n| \quad a x^2 \exp \qquad \frac{c_0 \log x}{(\log x_n)^{1}} + 2(\log x_n) \qquad a x^2 e^{\quad c(\log x)} \ , \quad 0 < \quad < 1, \qquad (4.20)$$

for some constants $a, c_0, c > 0$ and for sufficiently large n.

Our aim is to obtain large values for $2\Re(J_n)$ compared with the other error term of (4.13). For this purpose we recall equation (4.18)

$$J_n = J_n' + J_n'' = \quad \frac{\sin \frac{\quad n}{2} \quad x^{2 \quad a_n + i b_n}}{(\log x)^{\frac{\quad n}{2} + 1}} S_n + O \quad x^2 e^{\quad (\log x)^{1}} \qquad ;$$

We can rewrite the above equation as follows,

$$A = \frac{J_n}{x^{2 \quad a_n}} = B x^{i b_n} + C;$$

where $B = \quad \frac{\sin(\frac{\quad n}{2})}{} \quad \frac{1}{\log x}^{\frac{\quad n}{2} + 1} \quad \text{same} \quad 2(\log x)^{1} \qquad x$

From definition of B we have $\arg B = \arg S_n + \theta$. The main term (involving the function) on the right hand side of equation (4.19) is independent of r. Now, as r runs from $-\infty$ to $+\infty$, the argument of S_n (and therefore $\arg B$) does not exceed 2π, since the last two terms are much smaller than the first one. This tells us that, as r runs from $-\infty$ to $+\infty$,

$$b_n \log x_n + \arg B - \theta + b_n \log\left(1 + \frac{r}{\log x_n}\right);$$

runs through an interval centred somewhere in

$$\left[b_n \log x_n - 2\pi ; b_n \log x_n + 2\pi\right].$$

The highest point is at least

$$b_n \log x_n - 2\pi + b_n \log\left(1 + \frac{1}{\log x_n}\right);$$

whereas the lowest point is at most

$$b_n \log x_n + 2\pi + b_n \log\left(1 - \frac{1}{\log x_n}\right).$$

Therefore the length of (4.21) is

$$b_n \log\left(1 + \frac{1}{\log x_n}\right) - 4\pi \geq \frac{b_n}{1 + \log x_n} - 4\pi \to \infty; \text{ as } n \to \infty.$$

For large n we choose values (r^+ and r^-) of r appropriately such that

$$\Re\left(\frac{A - C}{|B|}\right) = +1 \text{ and } \Re\left(\frac{A - C}{|B|}\right) = -1.$$

For the first case we have $\Re(\frac{J_n}{x^{2-a_n}}) = \Re(A) = |B| + \Re(C)$; that is,

$$\Re(J_n) = |B| x^{2-a_n} + \Re(C) x^{2-a_n}$$

$$\geq |J_n| - |C| x^{2-a_n} + \Re(C) x^{2-a_n}$$

$$= |J_n| + O(x^2 e^{-(\log x)^{1-\epsilon}})$$

$$\geq A_0 x^2 e^{-c(\log x)}.$$

Therefore, for sufficiently large n we have

$$\Re(J_n) \geq A_0 x^2 e^{-c(\log x)}; \text{ for } r = r^+ \text{ and } A_0, c > 0. \tag{4.22}$$

Similarly we can get

$$<(J_n) \quad A_0 x^2 e^{\ c(\log x)} \ ; \text{ for } \ r = r \quad \text{and} \ A_0; c > 0: \tag{4.23}$$

From the above inequalities and the following equation

$$M_P(x) = \frac{x^2}{2} \, 2$$

4.2 Discrete g-prime System

In the above section, we found a continuous g-prime system for which $\ = \ .$ Now we show that it may be adapted to give a discrete version. Finding discrete system satisfying this same property is generally more challenging. The reason for this is that if we have $_P(x)$ defined as a step function, then seeing the singularities of the Beurling zeta function is difficult.

We shall use the method developed by Diamond, Montgomery, Vorhauer [11] and later Zhang [31] which uses (the theory of) probability measures to find discrete systems of Beurling primes.

Theorem 4.10. *Let* $0 < \ < 1$. *Then there is a discrete g-prime system* P *for which*

$$^d_P(x) = \mathrm{li}(x) + O(xe^{\ (\log x)\ }); \tag{4.26}$$

and

$$N^d_P(x) = \ x + \ (xe^{\ c(\log x)\ }); \tag{4.27}$$

for some positive constants $\ $ *and* c. *Thus* $(\ )$ $\ $ *for discrete systems.*

To find the g-prime satisfying (4.26) we use the following lemmas from Zhang's paper [31].

Lemma 1. *Let* $f(\ )$ *be a nonnegative-valued Lebesgue measurable function on* $(\ 1; 1)$ *with support* $[1; 1)$: *Assume that there is increasing function* $F(x)$ *on* $(\ 1; 1)$ *with support* $[1; 1)$ *satisfying*

$$\int_1^x f(\ )d\ \quad F(x);$$

$$\int_x^{x+1} f(\ )d\ \quad \sqrt{F(x)(1 + \log x)};$$

$$\log x = o(F(x));$$

$$\int_1^x \frac{1}{\sqrt{F(\ )}}d\ \quad \sqrt{F(x)};$$

and

$$F(x + 1) \quad F(x):$$

Let

$$1 \le \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \lambda_{k+1} < \cdots$$

be a sequence such that $\lambda_k \to \infty$ *as* $k \to \infty$ *and such that*

$$p_k = \int_{\lambda_{k-1}}^{\lambda_k} f(\lambda)\,d\lambda\,;$$

satisfies $0 < p_k < 1$ *for* $k > k_0$*. Then there is a subsequence* $\lambda_{k_j}; j = 1, 2, \ldots$ *such that*

$$\sum_{\lambda_{k_j} \le x} \lambda_{k_j}^{it} - \sum_{\lambda_k \le x} \lambda_k^{it} p_k \ll \sqrt{F(x)}\left(\sqrt{1 + \log x} + \sqrt{\log(t+1)}\right); \tag{4.28}$$

for $1 \le x < \infty$ *and* $t \ge 0$*.*

Lemma 3. *If the sequence* λ_k *in* **Lemma 1** *satisfies also*

$$\sum_{\lambda_k \le x} \lambda_k^{it} p_k - \int_1^x \lambda^{it} f(\lambda)\,d\lambda \ll \sqrt{F(x)}\left(\sqrt{1 + \log x} + \sqrt{\log(t+1)}\right); \tag{4.29}$$

for $F(x) \ge c \log(t+1)$ *with a constant* $c > 0$ *then there is a subsequence* $\lambda_{k_j}; j = 1, 2, \ldots$ *such that*

$$\sum_{\lambda_{k_j} \le x} \lambda_{k_j}^{it} - \int_1^x \lambda^{it} f(\lambda)\,d\lambda \ll \sqrt{F(x)}\left(\sqrt{1 + \log x} + \sqrt{\log(t+1)}\right); \tag{4.30}$$

for $1 \le x < \infty$ *and* $t \ge 0$*.*

Lemma 4. *Let* $f(x)$ *be a Lebesgue measurable function on* $(-\infty; \infty)$ *with support* $[1; \infty)$ *satisfying*

$$0 \le f(x) \le \frac{1 - x^{-1}}{\log x}.$$

Then the function

$$F(x) = \frac{x}{1 + \log x};$$

satisfies the conditions of **Lemma 1** *and both the sequences*

$$(1) \quad \lambda_k = \sqrt{\log(k + k_0)}; \; k = 0, 1, 2, \ldots$$

and

$$(2) \quad \lambda_k = \ldots \log 0$$

satisfy the conditions of **Lemma 1** *and* **Lemma 3**. *Therefore both* (1) *and* (2) *have a subsequence* γ_{k_j}; $j = 1, 2, \ldots$ *satisfying*

$$\left|\sum_{\gamma_{k_j}\le x}\gamma_{k_j}^{it}-\int_1^x \alpha^{it} f(\alpha)\,d\alpha\right| \ll \sqrt{x}\left(1+\sqrt{\frac{\log(t+1)}{1+\log x}}\right), \tag{4.31}$$

for $1 \le x < \infty$ *and* $t \ge 0$.

Now, consider the continuous function

$$h(\alpha) = \frac{1-\alpha^{-k}}{\log\alpha}\,\rho(\alpha); \quad \text{with} \quad \rho(\alpha) = 1 - \sum_{n>n_0}\epsilon_n \frac{\cos(b_n\log\alpha)}{a_n}; \qquad \alpha \ge 1.$$

That is, the function $h = \rho_P'$ where ρ_P from Theorem 4.1. Here k, n_0, ϵ_n, b_n and a_n as in Theorem 4.1. The function $h(\alpha) \ge \frac{1-\alpha^{-1}}{\log\alpha}$. So, by **Lemma 4** there is a sequence $1 \le \gamma_0 \le \gamma_1 \le \gamma_2 \le \cdots \le \gamma_j \le \gamma_{j+1} \le \cdots$ such that $\gamma_j \to \infty$ as $j \to \infty$ for which

$$\left|\sum_{\gamma_j\le x}\gamma_j^{it}-\int_1^x \alpha^{it} h(\alpha)\,d\alpha\right| \ll \sqrt{x}\left(1+\sqrt{\frac{\log(t+1)}{1+\log x}}\right),$$

for $1 \le x < \infty$ and $t \ge 0$. In particular, when $t = 0$ we have

$$\sum_{\gamma_j\le x} 1 - \int_1^x h(\alpha)\,d\alpha = O(\sqrt{x}). \tag{4.32}$$

We shall take $\{\gamma_j\}_{j\ge 0}$ as our g-primes. By Proposition 4.2 we get

$$\pi_P^d(x) := \sum_{\gamma_j\le x} 1 = \mathrm{li}(x) + O(xe^{-c(\log x)^{\theta}}) + O(\sqrt{x}) = \mathrm{li}(x) + O(xe^{-c(\log x)^{\theta}});$$

with θ as in section 4.1. We let

$$\Pi_P^d(x) = \sum_{n\ge 1}\frac{\pi_P^d(x^{\frac{1}{n}})}{n}.$$

Then

$$\Pi_P^d(x) = \mathrm{li}(x) + O(xe^{-c(\log x)^{\theta}}); \tag{4.33}$$

since $\Pi_P^d(x) = \pi_P^d(x) + O(\sqrt{x})$. This proves (4.26). We estimate $N_P^d(x)$ through its associated zeta function ζ_P^d given by

$$\begin{aligned}\zeta_P^d(s) &= \int_1^\infty x^{-s}\,dN_P^d(x) = \exp\left(\int_1^\infty x^{-s}\,d\Pi_P^d(x)\right)\\ &= \exp\left(-\int_1^\infty \log(1-x^{-s})\,d\pi_P^d(x)\right).\end{aligned} \tag{4.34}$$

This can be written as

$$ {}^{d}_{P}(s) = {}_{P}(s) \exp \{F_2(s) \quad F_1(s)\} ; \tag{4.35}$$

where ${}_{P}(s)$ as in Section 4.1 is analytic in D and

$$F_1(s) = \int_{1}^{1} \quad {}^{s} d \, {}^{d}_{P}(\;) + \log(1 \quad {}^{s}) d \, {}^{d}_{P}(\;) \quad ;$$

and

$$F_2(s) = \int_{1}^{1} \quad {}^{s} \; d \, {}^{d}_{P}(\;) \quad h(\;) d \quad :$$

We see $\log(1 \quad {}^{s}) = \quad {}^{s} + O(\quad {}^{2}$

Lemma 4.11. *Let a function $F(x;t)$ defined for $1 \le x < \infty$ and $t \ge 0$ be locally of bounded variation in x and satisfy $F(1;t) = 0$ and*

$$F(x;t) \ll \sqrt{x}\left(1 + \sqrt{\frac{\log(t+1)}{1+\log x}}\right).$$

Given $x \ge x_0 > 1$, let $\sigma \ge \frac{1}{2} + \delta$; $\delta > 0$. Then

$$\int_1^\infty x^{-\sigma}\, dF(x;t) \ll \sqrt{\log(t+1)}.$$

Proof. Using integration by parts, the integral on the left hand side is

$$\int_1^\infty \sigma x^{-\sigma-1} F(x;t)\, dx \ll \int_1^\infty x^{-\sigma-\frac{1}{2}}\left(1 + \sqrt{\frac{\log(t+1)}{1+\log x}}\right) dx \ll \sqrt{\log(t+1)},$$

since $\int_1^\infty x^{-\sigma-\frac{1}{2}}\, dx$ converges. □

Let

$$g(x;t) = \sum_{\lambda_j \le x} \lambda_j^{it} - \int_1^x u^{it} h(u)\, du; \quad x \ge 1.$$

So, $g(x;t)$ satisfies the conditions of Lemma 4.11. Thus, by Lemma 4.11, we have

$$F_2(\sigma + it) = \int_1^\infty u^{-\sigma}\left(u^{-it}\, dg(u;0)\right) \ll \sqrt{\log t}; \quad t \ge 2, \tag{4.37}$$

for $\sigma \ge \frac{1}{2} + \delta$; $\delta > 0$.

We shall need to use $T_n = \exp\{f(\log x_n)\}$, such that $0 < \alpha < \beta < 1$. Therefore,

$$F_2(\sigma + iT_n) = O((\log x_n)^{\frac{\beta}{2}}). \tag{4.38}$$

Also,

$$\begin{aligned}
F_1(s) &= \int_1^\infty -u^{-s}\, d\pi_P^d(u) + \log(1-u^{-s})\, d\pi_P^d(u) \\
&= -\int_1^\infty u^{-s}\, d\pi_P^d(u) + \sum_{m \ge 1} \frac{1}{m}\int_1^\infty u^{-ms}\, d\pi_P^d(u) \\
&= \sum_{m \ge 2} \frac{1}{m}\int_1^\infty u^{-ms}\, d\pi_P^d(u).
\end{aligned}$$

This shows that the integral for $F_1(s)$ converges unifomly for $\sigma \ge \frac{1}{2} + \delta$ with each $\delta > 0$. Therefore,

$$F_1(s) = O(1); \quad \text{for } \sigma \ge \frac{1}{2} + \delta; \quad \delta > 0. \tag{4.39}$$

Hence, we see from equations (4.38) and (4.39) that for $s = \quad + iT_n;$ we have

$$< fF_2(s) \quad F_1(s)g \quad jF_2(s) \quad F_1(s)j \quad b(\log x_n)^{\frac{}{2}}; \; b > 0: \tag{4.40}$$

This tells us that

$$< fF_2(s) \quad F_1(s)g \qquad b(\log x_n)^{\frac{}{2}}; \;\; \text{for some } b > 0: \tag{4.41}$$

From (4.37) and (4.39) we have proved the following

Corollary 4.12. *For $\quad + it \; 2 \; D \;\; \backslash \; H_{\frac{1}{2}}$; we have*

$$\overset{d}{P}$$

From this we see equation (4.42) becomes

$$M_P^d(x) = \frac{x^2}{2} + \sum_{n_0<|m|\ \ n\ \ 1} J_m^d + \{J^d\ _n + J_n^d\} + O\left(x^2 e^{\ c(\log x)^{\frac{2}{\ +1}}}\right); \quad > 0. \quad (4.43)$$

To estimate the second term in the right hand side of equation (4.43) we need to prove again Proposition 4.6 with $^d_P(s)$ instead of $_P(s)$ as follows

Proposition 4.13.

$$\sum_{n_0<|m|\ \ n\ \ 1} J_m^d = O\left(x^2 e^{\ q(\log x)^{1\ \frac{1}{\ }}}\right); \quad \textit{for some } q > 0.$$

Proof. Let us consider the integral J_m and let $_m$

We see that $1 \quad \frac{1}{\omega} \quad \frac{2}{+1} =$ since ω is taken sufficiently large, so equation (4.43) becomes

$$M_P^d($$

where

$$g_n(s) = e^{F_2(s) \quad F_1(s)} f_n(s):$$

Here $g_n(s)$ is analytic in a disc around the point $z_n = 1 \quad a_n + ib_n$. Therefore By Proposition 4.7 and equation (4.40) we obtain

$$jg_n(s)j \quad \frac{64}{b_n^2} \exp f<(F_2(s) \quad F_1(s))g \quad \frac{q_1}{b_n^2} e^{(\log x_n)^{\frac{}{2}}}; \qquad (4.46)$$

for some $q_1 > 0$. While, by Proposition 4.9 and equation (4.41) we have

$$jg_n(s)j \quad \frac{1}{160 b_n^2} \exp f<(F_2(s) \quad F_1(s))g \quad \frac{q_2}{b_n^2} e^{\quad (\log x_n)^{\frac{}{2}}}; \qquad (4.47)$$

for some $q_2 > 0$. Therefore, by (4.18) with $g_n(s)$ instead of $f_n(s)$ and by (4.47) we obtain

$$S_n^d \quad b \exp \quad (\log x_n)^{\frac{}{2}} + 2(\log x_n) \quad ; \quad b > 0; \qquad (4.48)$$

for x is su ciently large, that is for n is su ciently large. We use this lower bound of the integral S_n^d: Now considering the other factor in (4.17) (with S_n is replaced by S_n^d), we have

$$\frac{\sin \frac{n}{2}}{} x^{2 \quad a_n +} \quad \frac{1}{\log x}^{\frac{n}{2}+1} \quad \frac{n}{} x^2 e^{\quad a_n \log x} \quad \frac{1}{2(\log x)^2}$$

$$q_3 x^2 e^{\frac{\log x}{(\log x_n)^1}} \quad \frac{1}{(\log x)^2 (\log\log x_n)^2}; \quad q_3 > 0:$$

where $_n = \frac{1}{n^2}$ and $n \quad \frac{\log\log x_n}{! \quad 1}$: From the above and (4.48), we get

$$J_n^d \quad \frac{q_3 x^2}{(\log x)^2 (\log\log x_n)^2} \exp \quad \frac{\log x}{(\log x_n)^1} \quad 1$$

1x ; ; (4.46)for some q
83 013 Td [(2(log)]TJ/F23 11.9552 Tf 27.518 0 Td [(x)]TJ/F15 11.9552 Tf 6.652 0 Td [()]TJ/F21 7.9701 Tf 4.553 3.454 Td [(2)]T
052 Tf 19.876 9.683 Td9TJ/F2[]0 d 0<50

We next aim to obtain a large value for $J_n^d + J^d{}_n = 2<(J_n^d)$ compared with the other error term of (4.50). To achieve this we can use similar arguments as those

Chapter 5

Connecting the error term of $N_P(x)$ and the size of $\zeta_P(s)$

When proving results linking the asymptotic behaviour of $\pi_P(x)$ and $N_P(x)$ one often uses as a go-between the Beurling zeta function $\zeta_P(s)$. Thus an assumption made on $\pi_P(x)$ is translated into a property of $\zeta_P(s)$ which is then shown to imply a property of $N_P(x)$ and similarly vice versa. The property on $\zeta_P(s)$ is often related to its size along the vertical line (or holomorphicity). For example, if $N_P(x) = cx + O(x^\theta)$; $\theta < 1$. Then $\zeta_P(s)$ is holomorphic in $H_\theta \setminus \{1\}$ and $\zeta_P(\sigma + it) = O(t)$ for $\sigma > \theta$. That is, ζ_P has at most polynomial growth on vertical lines to the left of 1. Furthermore, bounds on the vertical growth can be shown via the inverse Mellin transform to imply $N_P(x) = cx + O(x^\theta)$. Here we investigate the connection when $N_P(x) = cx + \Omega(x^{1-\epsilon})$; and where $\zeta_P(\sigma + it)$ may have infinite order. Therefore, if we assume that $\zeta_P(s)$ has polynomial growth along some curve for $\sigma < 1$, what can be said about the behaviour of $N_P(x)$ (as $x \to \infty$) and vice versa?

We concentrate in this chapter on determining the connections between the asymptotic behaviour of the g-integer counting function $N_P(x)$ and the size of Beurling zeta function $\zeta_P(\sigma + it)$ with σ near 1 (as $t \to \infty$). We aim to find this link and apply it in chapter 6.

5.1 From ζ_P to N_P

We start with showing how assumptions on growth of $\zeta_P(s)$ imply estimates on the error term of $N_P(x)$. Note that in fact, the following theorem is purely analytical as there is no use of g-prime systems (only the fact that $N_P \in S_1^+$).

Theorem 5.1. *Suppose that for some $\theta \in [0,1)$, $\zeta_P(s)$ has an analytic continuation to the half plane H_θ except for a simple pole at $s = 1$ with residue ρ.*

Further assume that for some $c < 1$,

$$\zeta_P(\sigma + it) = O(t^c), \quad \text{for} \quad \sigma \geq 1 - \frac{1}{f(\log t)},$$

where f is a positive, strictly increasing continuous function, tending to infinity. Then for $\alpha = 1 - c$,

$$N_P(x) = \rho x + O(xe^{-\frac{\alpha}{2}h^{-1}(\alpha^{-1}\log x)}),$$

where $h(u) = uf(u)$.

Proof. We use the bound $\zeta_P(s) = O(t^c)$, for some $c < 1$ to find an approximate formula for

$$M_P(x) = \int_0^x N_P(y)dy = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \zeta_P(s)\frac{x^{s+1}}{s(s+1)}ds.$$

This holds for any $b > 1$. Pushing the contour to the left of the line $\Re s = b$ past the simple pole at 1, we get for any $T > 0$

$$M_P(x) = \frac{\rho}{2}x^2 + \frac{1}{2\pi i}\int_{\Gamma_T} \zeta_P(s)\frac{x^{s+1}}{s(s+1)}ds + \frac{1}{2\pi i}\int_{1-\frac{1}{f(\log T)}+iT}^{b+iT} \zeta_P(s)\frac{x^{s+1}}{s(s+1)}ds$$
$$+ \frac{1}{2\pi i}\int_{b-iT}^{1-\frac{1}{f(\log T)}-iT} \zeta_P(s)\frac{x^{s+1}}{s(s+1)}ds + \frac{1}{2\pi i}\left(\int_{b+iT}^{b+i\infty} + \int_{b-i\infty}^{b-iT}\right) \zeta_P(s)\frac{x^{s+1}}{s(s+1)}ds.$$

Here Γ_T is the contour $s = 1 - \frac{1}{f(\log t)} + it$ for $a < |t| \leq T$ and $s = 1 - \frac{1}{f(\log a)} + it$ for $|t| \leq a$. The constant a is chosen such that $a > e$ and $1 - \frac{1}{f(\log a)} > \theta$, (see Figure 5.1).

The modulus of the integral over the horizontal line $[1 \frac{1}{f(\log T)} + iT; b + iT]$ is

$$\int_{1 \frac{1}{f(\log T)} + iT}^{b+iT} {}_{P}(s)\frac{x^{s+1}}{s(s+1)}ds = O \int_{1 \frac{1}{f(\log T)}}^{b} \frac{T^c x^{u+1}}{T^2}du$$

$$= O(\quad x$$

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$$= O \quad x^2 \int_{\log a}^{\infty} \exp \quad (2u + \frac{\log x}{f(u)}) + (c+1)u \quad du \quad + O(x^2 \quad \frac{exp \; 1}{f(\log a)});$$

$$= O \quad x^2 \int_{\log a}^{\infty} \exp \quad ((1 \quad c)u + \frac{\log x}{f(u)}) \quad du \quad + O(x^2 \quad \frac{1}{f(\log a)}):$$

To estimate the integral, we split it into

$$\int_{\log a}^{\infty} \exp \quad (\quad u + \frac{\log x}{f(u)}) \quad du = \int_{\log a}^{A} + \int_{A}^{\infty} \quad \exp \quad (\quad u + \frac{\log x}{f(u)}) \quad du;$$

for some $A > \log a$ and $\quad = 1 \quad c:$

The rst integral over $(\log a; A)$ is $e^{\frac{\log x}{f(A)}} \int_{\log a}^{A} e^{\quad u} du = O(e^{\frac{\log x}{f(A)}});$ whilst

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Similarly, the right hand side of the same inequality is

$$= \frac{1}{y} \quad xy + \frac{y^2}{2} + O \;\; x^2 \exp f \quad h^{\;1}(\quad ^{1} \log x) g \quad ;$$

hold for any $0 < y < x$.

Now, for some $\quad > 0$ and some $d > 0$ we have $h(x) \quad h(x \quad d) = xf(x) \quad (x \quad d)f(x \quad d) = x \;\; f(x) \quad f(x \quad d) \;\; + df(x \quad d) \qquad > 0:$

2. For $f(x) = x$ for some > 0. We have $h(x) = x^{1+}$ and

$$h\ ^{1}(\log x) = (\log x)^{\frac{1}{1+}}:$$

That is,

$$_P(1 \quad \frac{1}{(\log t)} + it) = O(t^c); \quad (c < 1) \quad \text{implies}$$

$$N_P(x) = \quad x + O \quad x \exp f \quad b(\log x)^{\frac{1}{1+}} g \quad ;$$

for some $> 0;$ where $b = \frac{\overline{1+}}{2}$ and $= 1 \quad c:$

5.2 From N_P to polynomial growth of ζ_P

Our purpose in this section is to obtain a kind of converse of Theorem 5.1. That is, we find the region where $\zeta_P(\sigma + it) = O(t^c)$; for some $c > 0$, if we assume that we have a bound for the error term of $N_P(x)$. In the other words, the reason of the following theorem is to obtain polynomial growth for $\zeta_P(\sigma + it)$, $\sigma < 1$. This depends on θ and the bound of the error term of $N_P(x)$. We shall need to assume a priori that ζ_P is has an analytic continuation to the left of $\sigma = 1$ and that $\zeta_P(\sigma + it)$ is bounded above by $O(e^t)$ for $0 \le \sigma < 1$.

Theorem 5.2. *Suppose that for some $\theta \in [0, 1)$; $\zeta_P(s)$ has an analytic continuation to the half plane H_θ except for a simple pole at $s = 1$ with residue κ and for $\sigma > \theta$; $\zeta_P(\sigma + it) = O(e^t)$; $(t > 0, t \to \infty)$.*

Further assume that

$$N_P(x) = \kappa x + O(xe^{-k(x)}),$$

for some positive, increasing function k tending to infinity such that $k'(x) = o(\frac{1}{x})$. Then for some $c > 0$,

$$\zeta_P(\sigma + it) = O(t^c),$$

for $1 - \frac{k(\frac{e^t}{t})}{t} \le \sigma < 1 - \frac{\log t}{t}$, where t is sufficiently large.

Proof. The usual *Mellin transform*

$$\zeta_P(s) = \int_1^\infty x^{-s} dN_P(x), \quad \sigma > 1$$

cannot be used directly for $\sigma < 1$, since the error term is not small enough to ensure analytic continuation to $\sigma < 1$. Instead we use a formula which is based on:

$$\sum_{n \ge 1} \frac{a_n}{n^s} e^{-(\delta n)} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty}$$

We generalize equation (5.3) (with $\lambda = 1$) in terms of the Beurling zeta function. The reason for doing this is to find an estimate for $\zeta_P(s)$ for $\sigma < 1$. That is, we show

$$\int_1^\infty x^{-s}e^{-\lambda x}dN_P(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\omega)\,\zeta_P(s+\omega)\,\lambda^{-\omega}d\omega; \qquad (5.4)$$

holds for $\lambda > 0$ and $c > 0$, $c > 1-\sigma$.

To see this, notice that the right hand side of equation (5.4) is equal

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\omega)\left(\int_1^\infty x^{-(s+\omega)}dN_P(x)\right)\lambda^{-\omega}d\omega;$$

and observe that we can invert the order of integrations by 'absolute convergence' since gamma is exponentially small. It becomes

$$\int_1^\infty x^{-s}\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\omega)(\lambda x)^{-\omega}d\omega\right)dN_P(x) = \int_1^\infty x^{-s}e^{-\lambda x}dN_P(x).$$

Note that both sides of (5.4) are entire functions.

Now we integrate by parts the left hand side of equation (5.4) and on the right we push the contour to the left of the lines $\Re\omega = 0$ and $\Re\omega = 1-\sigma$. We get

$$\lambda\int_1^\infty e^{-\lambda x}x^{-s}N_P(x)dx + s\int_1^\infty e^{-\lambda x}x^{-s-1}N_P(x)dx = \zeta_P(s) + \lambda^{s-1}\Gamma(1-s)$$
$$+\frac{1}{2\pi i}\left(\int_{c-i\infty}^{c'-i\infty}+\int_{c'-i\infty}^{c'+i\infty}+\int_{c'+i\infty}^{c+i\infty}\right)\Gamma(\omega)\,\zeta_P(s+\omega)\,\lambda^{-\omega}d\omega;$$

for some negative constant $c' > -1$ and $c'+\sigma > \theta$, since the integrand of the right hand side of equation (5.4) has singularities at $\omega = 0$ and $\omega = 1-s$ with residues $\zeta_P(s)$ and $\lambda^{s-1}\Gamma(1-s)$ respectively. The contribution from the horizontal line $[c'+iy, c+iy]$ is

$$\int_{c'+iy}^{c+iy}\Gamma(\omega)\,\zeta_P(s+\omega)\,\lambda^{-\omega}d\omega = \int_{c'}^{c}\Gamma(x+iy)\,\zeta_P(\sigma+x+i(y+t))\,\lambda^{-(x+iy)}dx$$
$$= O\left(|y|^{-\frac12}\exp\left(\epsilon|y+t| - \frac{\pi|y|}{2}\right)\int_{c'}^{c}y^x\lambda^{-x}dx\right) \to 0 \text{ as } y\to\infty.$$

since $|\Gamma(x+iy)|$

As we mentioned earlier, we are interested in finding an estimate for $\zeta_P(s)$ for $\sigma < 1$. So, we take $\delta < \sigma < 1$; and try to estimate each term in the right hand side of equation (5.5) separately. For the first integral we have

$$\int_1^\infty e^{-\lambda x} x^{-s} N_P(x)dx = \int_1^\infty e^{-\lambda x} x^{-s}\left(x + O(xe^{-k(x)})\right)dx$$

$$= \int_1^\infty e^{-\lambda x} x^{-s+1}dx + O\left(\int_1^\infty x^{-\sigma+1}e^{-(\lambda x+k(x))}dx\right).$$

$$= \lambda^{s-1}\Gamma(2-s) - \int_0^1 e^{-\lambda x} x^{-s+1}dx + O\left(\int_1^\infty x^{-\sigma+1}e^{-(\lambda x+k(x))}dx\right);$$

since $\sigma < 1$. Hence

$$\int_1^\infty e^{-\lambda x} x^{-s} N_P(x)dx = \lambda^{s-1}\Gamma(2-s) + O\left(\int_1^\infty x^{-\sigma+1}e^{-(\lambda x+k(x))}dx\right) + O(1); \tag{5.6}$$

since $\int_0^1 e^{-\lambda x} x^{-s+1}dx = O(1)$.

For the second integral of equation (5.5) we have

$$s\int_1^\infty e^{-\lambda x} x^{-s-1}N_P(x)d = s\int_1^\infty e^{-\lambda x} x^{-s-1}\left(x + O(xe^{-k(x)})\right)dx$$

$$= s\int_1^\infty e^{-\lambda x} x^{-s}dx + O\left(t\int_1^\infty x^{-\sigma} e^{-(\lambda x+k(x))}dx\right);$$

$$= s\lambda^{s-1}\Gamma(1-s) - s\int_0^1 e^{-\lambda x} x^{-s}dx + O\left(t\int_1^\infty x^{-\sigma} e^{-(\lambda x+k(x))}dx\right);$$

since $\sigma < 1$. Hence

$$s\int_1^\infty e^{-\lambda x} x^{-s-1}N_P(x)dx = s\lambda^{s-1}\Gamma(1-s)+O\left(\frac{t}{1-\sigma}\right)+O\left(t\int_1^\infty x^{-\sigma} e^{-(\lambda x+k(x))}dx\right); \tag{5.7}$$

since $\left|s\int_0^1 e^{-\lambda x} x^{-s}dx\right| = O(t\int_0^1 x^{-\sigma}dx) = O(\frac{t}{1-\sigma})$.

Finally, for the vertical line over $[c' - i\infty, c' + i\infty]$ we have

$$\int_{c'-i\infty}^{c'+i\infty} \Gamma(\omega)\zeta_P(s+\omega)\lambda^{-\omega}d\omega = \int_{-\infty}^{\infty} \Gamma(c'+iy)\zeta_P(\sigma + c' + i(y+t))\lambda^{-(c'+iy)}dy$$

$$= O\left(\lambda^{-c'}\int_{-\infty}^{\infty}(|y|+1)^{c'-\frac{1}{2}}\exp\left(|y+t| - \frac{|y|}{2}\right)dy\right)$$

$$= O\left(\lambda^{-c'}e^{t}\right).$$

From the above, equation (5.5) becomes

$$\begin{aligned}
\zeta_P(s) &= \alpha^{s-1}\Gamma(2-s) + s\,\alpha^{s-1}\Gamma(1-s) - \alpha^{s-1}\Gamma(1-s) + O\Big(\frac{t}{1-\sigma}\Big) + O\big(\alpha^{c'}e^{t}\big)\\
&\quad + O\Big(\int_1^\infty x^{-\sigma+1}e^{-(\alpha x+k(x))}dx\Big) + O\Big(t\int_1^\infty x^{-\sigma}e^{-(\alpha x+k(x))}dx\Big);\\
&= O\Big(\int_1^\infty x^{-\sigma+1}e^{-(\alpha x+k(x))}dx\Big) + O\Big(t\int_1^\infty x^{-\sigma}e^{-(\alpha x+k(x))}dx\Big)\\
&\qquad + O\Big(\frac{t}{1-\sigma}\Big) + O\big(\alpha^{c'}e^{t}\big);
\end{aligned} \tag{5.8}$$

since for $\sigma < 1$, the gamma terms cancel each other.

Our aim is to find α for which we have $\zeta_P(s) = O(t^c)$ for some $c > 0$. So, putting $\alpha = e^{-t}$, we see that the last term of the right hand side of equation (5.8) is $O(1)$, since $c' > 1$. We see also with $\alpha = e^{-t}$ that the term

$$\int_1^\infty x^{-\sigma+1}e^{-(\alpha x+k(x))}dx = O(t).$$

Indeed, we split the integral into the ranges $(1, B)$ and (B, ∞) for some $B > 1$. For the first integral we have

$$\begin{aligned}
\int_1^B x^{1-\sigma}e^{-(\alpha x+k(x))}dx &\le B^{1-\sigma}\int_1^B e^{-k(x)}dx\\
&\le B^{1-\sigma}\Big(Be^{-k(B)} + \int_1^B xk'(x)e^{-k(x)}dx\Big) \ll B^{2-\sigma}e^{-k(B)};
\end{aligned}$$

since $k'(x) = o(\frac{1}{x})$. For the range (B, ∞) the second integral is

$$\begin{aligned}
\int_B^\infty x^{1-\sigma}e^{-(\alpha x+k(x))}dx &\le e^{-k(B)}\int_B^\infty x^{1-\sigma}e^{-\alpha x}dx\\
&= \alpha^{\sigma-2}e^{-k(B)}\int_{\alpha B}^\infty y^{1-\sigma}e^{-y}dy\\
&\le \alpha^{\sigma-2}e^{-k(B)}\Gamma(2-\sigma).
\end{aligned}$$

Therefore,

$$\int_1^\infty x^{1-\sigma}e^{-(\alpha x+k(x))}dx \ll B^{2-\sigma}e^{-k(B)} + \alpha^{\sigma-1}e^{-k(B)}\Gamma(2-\sigma).$$

Now, setting $B = \sqrt{\frac{1}{\alpha}}$, we get

$$\int_1^\infty x^{1-\sigma}e^{-(\alpha x+k(x))}dx \ll \alpha^{\frac{\sigma-2}{2}}e^{-k(\sqrt{\frac{1}{\alpha}})} + \alpha^{\sigma-1}e^{-k(\sqrt{\frac{1}{\alpha}})}\Gamma(2-\sigma).$$

For $1 \quad > \frac{\log t}{t}$ we see $\quad ^{1} < \quad \frac{\log t}{t} = t$. Therefore, the rst term in the right hand side of the above inequality $\to 0$ as $t \to \infty$, whilst the second term is $O(t)$. Thus, equation (5.8) becomes

$$_P(s) = O\left(t \int_1^{\infty} x \quad e^{(\ x + k(x))} dx \right) + O\left(\frac{t}{1}\right): \tag{5.9}$$

To estimate the rst integral in the right hand side of equation (5.9) we split it into the ranges $(1; A)$ and $(A; \infty)$ for some $A > 1:$

The rst integral is less than

$$\int_1^A \frac{e^{\ k(x)}}{x} dx \quad \frac{A^1 \quad e^{\ k(A)}}{1} + \frac{1}{1} \int_1^A \frac{x k'(x)}{x} e^{\ k(x)} dx \quad \frac{A^1 \quad e^{\ k(A)}}{1};$$

since $k'(x) = o(\frac{1}{x});$ whilst the second integral over $(A; \infty)$ is

$$\frac{e^{\ k(A)}}{A} \int_A^{\infty} e^{\ x} dx \quad \frac{e^{\ k(A)}}{A}:$$

So these tell us that

$$\int_1^{\infty} x \quad e^{(\ x + k(x))} dx \quad \frac{A^1 \quad e^{\ k(A)}}{1} + \frac{e^t \ k(A)}{A} = \frac{e^{\ k(A)}}{A} \quad \frac{A}{1} + e^t \quad :$$

Choose $A = (1 \quad)e^t$, equation (5.9) becomes

$$\begin{aligned} _P(s) &= O\left(\frac{t}{1} \quad (1 \quad)e^{t \ 1} \quad e^{\ k((1 \quad)e^t)} \right) + O\left(\frac{t}{1}\right) \\ &= O\left(t^2 \exp \quad t(1 \quad) \quad k((1 \quad)e^t) \right) + O(t^2); \end{aligned} \tag{5.10}$$

since $\frac{1}{1} < \frac{t}{\log t};$ and $(1 \quad)^1 \quad \to 1$ as $1 \quad \to 0:$ Therefore

$$_P(\ + it) = O(t^c) \quad \text{for some } c > 0; \tag{5.11}$$

when

$$\exp \quad t(1 \quad \mathrm{O}(t \quad)e^t \quad \mathrm{t}(1$$

since 1 $> \frac{\log t}{t} > \frac{1}{t}:$ This shows that (5.11) holds when

$$1 \qquad \frac{k \, \frac{e^t}{t}}{t}:$$

Therefore, for

$$1 \quad \frac{k \, \frac{e^t}{t}}{t} \qquad < 1 \quad \frac{\log t}{t};$$

we have $_P(s) = O(t^c)$ for some $c > 0$. □

We now illustrate Theorem 5.2 with some examples (of course, in each case we assume that $_P$ has an analytic continuation to H).

Examples

1. For $k(x) = (\log x)$, for some $2\ (0; 1):$ This means $k \ \frac{e^x}{x}$:

Chapter 6

Application to a particular example.

In this chapter we investigate a particular example of a g-prime system P_0. In this example, $_P$ is given explicitly (see De nition 15) and hence gives iv1and hence

Here $E(x) = o(x).$ We find O results and results for $E(x)$ as an application of Theorem 5.1 and Theorem 5.2.

Now, equation (6.1) (which implies ${}_0(x) = x + O(1)$) tells us that ${}_0(s)$ has an analytic continuation to the half plane $\{s \in \mathbb{C} : \Re s > 0\}$ except for a simple pole at $s = 1$ and ${}_0(s) \neq 0$ in this region (see Lemma 3.2). Moreover,

$$-\frac{{}_0'}{{}_0}(s) = \int_1^\infty x^{-s} d\,{}_0(x) = (s) - 1. \tag{6.3}$$

Here, the

for some $c > 0$.

We can improve on this by using Theorem 5.1. The real reason which allows us to improve on (6.5) is that $_0(s)$ is connected to the Riemann zeta function and we can use all the available information on (s).

Theorem 6.1. *We have*

$$N_0(x) = x + O \quad x \exp f \quad b(\log x)^{\frac{3}{5}} \quad \log\log x^{\frac{2}{5}} g \quad ; \tag{6.6}$$

for some $b > 0$. Furthermore, on the Riemann Hypothesis this can be improved to

$$N_0(x) = x + O \quad x \exp \quad \frac{(1 \quad) \log x \log\log\log x}{4 \log\log x} \quad ; \quad \textit{for every} \quad > 0: \tag{6.7}$$

Proof. Firstly, we show that 7ogWeve

(s) (1-222[(+)]TJ/F23 11.9552 Tf6248273 0 Td [tN

: (68();)]T9 -69.4022 - 7.020 Td [Nho,e for

$(s)s$ (s)

Our aim here is to apply Theorem 5.1. For this purpose we have to show for which region (near 1), $_0(+ it)$ t^c for some positive constant $c < 1.$ So, in order for $j\ _0(+ it)j$ $t^c;$ to hold for some $c < 1;$ we need

$$\exp \ (1 + t^{100(1 \quad)^{\frac{3}{2}}})(\log t)^{\frac{2}{3}} \qquad t^c.$$

That is, we need

$$1 + e^{100(1 \quad)^{\frac{3}{2}} \log t} \qquad c(\log t)^{\frac{1}{3}}.$$

This certainly holds for t su ciently large if

$$100(1 \qquad)^{\frac{3}{2}} \log t \qquad \frac{1}{4} \log \log t.$$

Therefore, for

$$1 \qquad \frac{\log \log t}{400 \log t}^{\frac{2}{3}};$$

we have

$$_0(+ it) = O(t^c); \quad \text{for some positive constant } c < 1.$$

Thus, we can apply Theorem 5.1. We have $f(x) = (\frac{400x}{\log x})^{\frac{2}{3}}$, which tells us that $h(x$

for some $A; a > 0$.

However, $\frac{e^u - 1}{u} \quad e^u$ for all $u \quad 0$. Therefore, in order for $\log j \ _0(\quad + it)j$ $c \log t;$ for some positive constant $c < 1;$ it is su cient to have

$$\exp \quad a(\log t)^{2(1 \quad)} + a \log \log \log t \qquad A_1 \log t;$$

for some $A_1; a > 0:$ That is,

$$a(\log t)^{2(1 \quad)} \quad \log \log t + \log A_1 \quad a \log \log \log t:$$

So, for $\quad 1 \quad \frac{\log \log \log t \quad k_1}{2 \log \log t};$ the above holds for some suitable $k_1 > 0;$ if t is su ciently large. That is, in this region we have

$$_0(\quad + it) = O(t^c); \quad \text{for every} \quad c > 0:$$

Now, apply Theorem 5.1 with $f(x) = \frac{2 \log x}{\log \log x \quad k_1}$

Proof. If (6.11) is not true then $N_0(x) = \ \ x + o(x^1 \ \ )$; which implies that $N_0(x) = \ \ x + O(x^1 \ \ )$; for some $\ \ > 0$: Thus we have a `well-behaved' system (see section 3).

$$_0(x) = x + O(1)$$

$$N_0(x) = \ \ x + O(x^1 \ \ ); \qquad > 0;$$

By Lemma 3.3 for $1 \qquad < \quad < 1$; we have

$$(s) \quad 1 = \quad {}^{0}_{0}$$

Proof of Theorem 6.3. If (6.12) is not true then

$$N_0(x) = x + O(xe^{-ck(x)}) \quad \text{for some } c > 1.$$

We know that $\zeta_0(s)$ has an analytic continuation to $\mathbb{C}\setminus\{1\}$ with a simple pole at $s = 1$ and $\zeta_0(s) \neq 0$ in this region. Moreover, by (6.10) we have $\zeta_0(\sigma + it) \ll e^{\epsilon t}$; $\forall \epsilon > 0$ for $\sigma_0 \le \sigma < 1$; (some fixed σ_0). Therefore, we can apply Theorem 5.2 as the conditions are satisfied. We obtain

$$\zeta_0(\sigma + it) = O(t^b); \quad \text{for some } b > 0;$$

for $1 - \frac{\log t}{t} \le \sigma \le 1 - \frac{(1-\delta)c\log\log\log t}{\log\log t}$, (any $\delta > 0$, and $t > t_0(\delta)$ since $\frac{ck(\frac{e^t}{t})}{t} \sim \frac{c\log\log\log t}{\log\log t}$). Furthermore, (6.9) tells us that

$$\log|\zeta_0(s)| \le \int_\sigma^2 |\zeta(u + it) - 1|\,du + O(1) \ll \log t;$$

since $\zeta(s) = O(\log t)$ for $1 - \frac{a}{\log t} \le \sigma \le 2$, (any $a > 0$), see Theorem 3.5. in [29]. Let $B(t) = \frac{(1-\delta)c\log\log\log t}{\log\log t}.$ Therefore, for $1 - B(t) \le \sigma \le 2$,

$$\log|\zeta_0(\sigma + it)| \le A\log t \quad \text{for some } A > 0.$$

Consider concentric circles with centre $\vartheta + it$ (for some $\vartheta > 1$) and radii $R_1 = \vartheta - 1 + B(t) - \eta(t)$ and $R_2 = \vartheta - 1 + B(t) - 2\eta(t)$; (with $\eta(t) = \frac{1}{\log\log t}$). Apply the Borel-Caratheodory Theorem to $\log\zeta_0(z)$; (see 9.1 in [29]). Therefore, for $\sigma \ge 1 - B(t) + \eta(t)$ and $t \ge t_0$; we obtain

$$\begin{aligned}|\log\zeta_0(\sigma + it)| &\le \frac{2R_2}{R_1 - R_2}A\log t + \frac{R_1 + R_2}{R_1 - R_2}|\log\zeta_0(\vartheta + it)| \\ &\le \frac{A_1}{\eta(t)}\log t + \frac{D}{\eta(t)} \ll \log t\log\log t;\end{aligned}$$

for some $A_1, D > 0.$

Now, let C be the circle (see Figure 6.1) with centre $1 - dB(t) + \eta(t) + it$; for some $d \in (\frac{1}{2}, 1)$ and radius $R = rB(t)$ for some positive constant $r < 1 - d.$ By Cauchy's integral formula

$$\frac{\zeta_0'}{\zeta_0}(s) = \frac{1}{2\pi i}\int_C \frac{\log\zeta_0(z)}{(z - s)^2}dz \quad \text{for } s \in C.$$

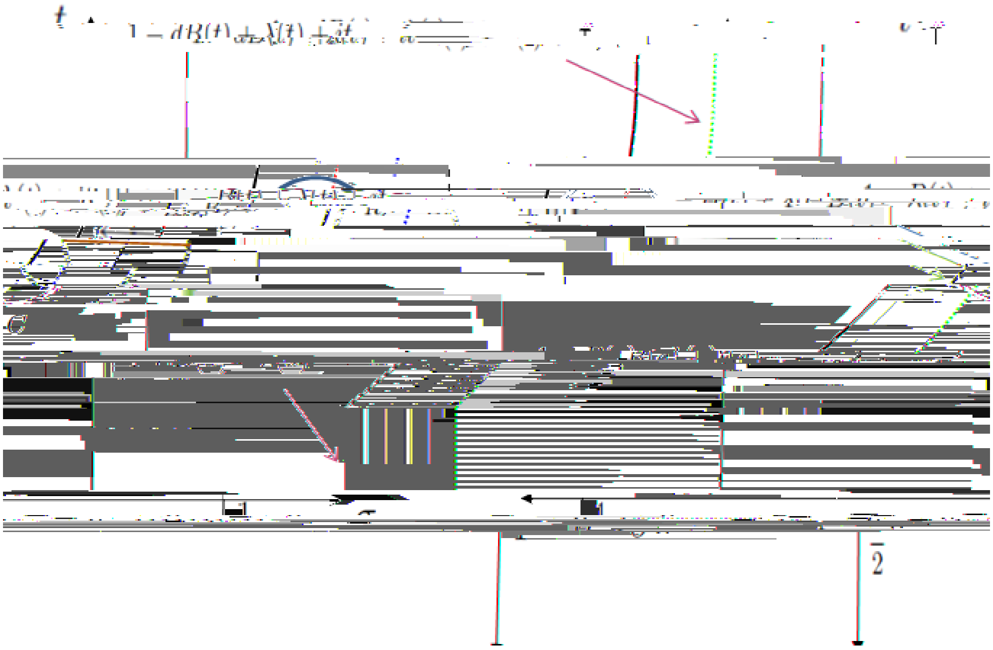

Figure 6.1: circle C

Therefore, for $s \in C$; we have

$$\frac{\zeta'}{\zeta}(s) \ll \frac{1}{R}\max_{z \text{ on } C} |\log \zeta_0(z)| \ll \frac{1}{B(t)} \log t \log\log t = o(\log^2 t).$$

So, this tells us that for $s \in C$

$$\frac{\zeta'}{\zeta}(s) = o(\log^2 t); \quad \text{for} \quad \sigma \geq 1 - \frac{(1-\epsilon)c\log\log\log t}{\log\log t} \tag{6.13}$$

However, by Proposition 6.4, for $1 - \sigma = \frac{(1-\epsilon)c\log\log\log T}{\log\log T}$; with $(1-\epsilon)c > 1$; we have

$$\begin{aligned}
\max_{1<t<T} \left|\frac{\zeta'}{\zeta}(\sigma + it)\right| &\geq \exp\left(\frac{\left(1+o(1)\right)(\log T)^{1-\sigma}}{16(1-\sigma)\log\log T}\right) \\
&= \exp\left(\left(1+o(1)\right)\frac{(\log\log T)^{(1-\epsilon)c}}{16(1-\epsilon)c\log\log\log T}\right) \\
&> e^{2\log\log T} = (\log T)^2.
\end{aligned}$$

This is a contradiction with (6.13). □

In the previous sections we have been trying to obtain good lower and upper bounds for $N_0(x) - x$. That is, upper bounds for $N_0(x) - x$ which holds for all sufficiently large values of x and lower bound for $N_0(x) - x$ which holds for a sequence of x's tending to infinity [and not necessarily for all (sufficiently large) values of x]. For the upper bound of $N_0(x) - x$ we have shown some unconditional O results and one result was conditional with the unproved Riemann Hypothesis.

Set $\Delta(x) = N_0(x) - x.$ A comparison of the O results and Ω results (based on Theorem 6.1 and Theorem 6.3) of this chapter, we have shown that

$$\Delta(x) = \Omega\left(x \exp\left(-\frac{c \log x \log\log\log\log x}{\log\log\log x}\right)\right) \quad \text{for every } c > 1;$$

while on the Riemann Hypothesis,

$$\Delta(x) \ll x \exp\left(-\frac{(1-\epsilon) \log x \log\log\log x}{4 \log\log x}\right); \quad \text{for every } \epsilon > 0.$$

This shows that there is a small gap between these results which reflects the great difficulty in determining the behaviours of $\zeta_0(s)$ in the strip $\frac{1}{2} < \sigma < 1.$ The interesting question is: What is the true order of this error term?

6.3 P_0 is a g-prime system

We end this chapter by showing that the pair (Π_0, N_0) is a g-prime system. That is, we show $\Pi_0 \in S_0^+$; (i.e. Π_0 is increasing).

Theorem 6.5. *(Π_0, N_0) is a g-prime system.*

We prove Theorem 6.5 by showing $\Pi_0 \in S_0^+$. Writing

$$\#_0(x) = \int_1^x \log y \, d\Pi_0(y), \tag{6.14}$$

which tells us that $\Pi_0 \in S_0^+ \iff \#_0 \in S_0^+$. Therefore, we will show that $\#_0 \in S_0^+$ and this will complete the proof of Theorem 6.5.

Now, we have

$$\Pi_0(x) = \sum_{n=1}^{\infty} \#_0(x^{\frac{1}{n}}),$$

(see definition 13 in chapter 3) and by the Möbius Inversion Formula we get

$$\#_0(x) = \sum_{n=1}^{\infty} \mu(n)\, \Pi_0(x^{\frac{1}{n}}).$$

Therefore, with $\Pi_0(x) = [x] - 1$, $x \geq 1$, we have

$$\#_0(x) = \sum_{n=1}^{\infty} \mu(n) \left([x^{\frac{1}{n}}] - 1 \right).$$

[**Note**: The above series is finite since the terms are zero for $n > \frac{\log x}{\log 2}$.] The following Proposition will complete the proof.

Proposition 6.6. *Let $\#_0(x) = \sum_{n=1}^{\infty} \mu(n) \left([x^{\frac{1}{n}}] - 1 \right)$, $x \geq 1$. Then the following hold:*

(i) $\#_0(x) = \#_0(k)$ *for* $k \leq x < k + 1$, $k \in \mathbb{N}$; (*i.e.* $\#_0(x) = \#_0([x])$.)

(ii) *Define* $f(n, k) = [k^{\frac{1}{n}}] - [(k-1)^{\frac{1}{n}}]$, $n, k \in \mathbb{N}$. *Then*

$$f(n, k) = \begin{cases} 1 & \text{if } k = q^n, \text{ for some } q \in \mathbb{N} \\ \end{cases}$$

(iii) *We call* $k \in \mathbb{N}$ *a* perfect power *if there exist natural numbers* $q > 1$, *and* $n > 1$ *such that* $k = q^n$: *Then*

$$\#_0(k) - \#_0(k-1) = \begin{cases} 1 & \textit{if } k \textit{ is not a perfect power }; \ k \geq 2 \\ 0 & \textit{if } k \textit{ is a perfect power }: \end{cases}$$

Proof. (i) For $k \leq x < k+1$; $k \in \mathbb{N}$

$$\#_0(x) - \#_0(k) = \sum_{n=1}^{\infty} (n) \ [x^{\frac{1}{n}}] - [k^{\frac{1}{-}}$$

However,

(iii) We have from (ii) that

$$f(n;k) = \begin{cases} 1 & \text{if } k = q^n; \quad \text{for some } \ q \in \mathbb{N}; \\ 0 & \text{if } k \neq q^n; \quad q \in \mathbb{N}. \end{cases}$$

Therefore, for k is not a perfect power, we have

$$\#_0(k) - \#_0(k-1) = \sum_{n=1}^{\infty} \mu(n) f(n;k) = 1 + \sum_{n=2}^{\infty} \mu(n) f(n;k) = 1.$$

Now, for k is a perfect power, let r be a maximal natural number greater than or equal 2 such that $k = q^r; \ q > 1; \ q \in \mathbb{N}.$ So, by the above Remark we have

$$\#_0(k) - \#_0(k-1) = \#_0(q^r) - \#_0(q^r - 1) = \sum_{n=1}^{\infty} \mu(n) f(n;q^r) = \sum_{n|r} \mu(n).$$

By Theorem 2.1 of [2] we have the last sum is zero (since $r > 1$). The proof of Proposition 6.6 is completed.

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