Analysis and Computation of a Simple Glacier Model using Moving Grids

Mathematical and Numerical Modelling of the Atmosphere and Oceans MSc 2009

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This dissertation is submitted to the Department of Mathematics and the Department of Meteorology in partial fulfilment of the requirements for the degree of Master of Science

Abstract

In this dissertation we are concerned with the study of glaciers, and analysing a simple model that determines the basis for the glacier to flow. We set up a PDE model with diffusion and source terms in one-dimension. From this model we define a nodal velocity associated with mass conservation, leading to the movement of a grid.

We assess the velocity using two methods, the first by assuming that a subdomain will hold the same properties as the whole domain, and the second by assuming the normalised ice volume remains constant in time. The analytical and computational benefits of each is considered.

Next we allow for surface elevation, using the subdomain assumption. In addition the velocity will be changed to allow for the effect of basal sliding. The velocity satisfies a Burgers-like equation, and the theory of characteristics and shocks is used to try and determine our aim of finding out the movement time.

Finally, consideration will be given to where this work can be taken next.

Acknowledgements

This year has reignited my passion for maths, and I have all the lecturers and my peers to thank for that! Also without funding from the NERC none of this would be possible so thank you to them. Finally special thanks to my supervisor Mike Baines, who is a complete legend and the most helpful guy imaginable!

Declaration

I con rm that this is my own work and the use of all materials from other sources have been properly and fully acknowledged.

Signed.................................. Date............................

Contents

List of Figures

Chapter 1

Introduction

Glaciers and Ice Sheets are a hot topic at the moment with global warming causing the considerable retreat of ice. Sometimes in order to understand for estimating the velocity on a moving grid will be analysed and computed, with results and comments as to the benefits of each.

Next we shall look at the impact of making the model more complex, with the addition of surface elevation and basal sliding, using the first velocity method.

In Chapter 7 we derive a Burgers-like equation for the velocity. We then examine the application of characteristics theory and the potential benefits such results could supply, with the aim of estimating a shock, leading to movement at the boundary of the glacier. Finally, consideration is given to ways the model may be extended, and the impacts these extensions may have on the results we have concluded, leading the way to potential further work to be undertaken in the topic.

For most of the project we are not concerned with physical values for the variables, but more with the methodology and the theory behind why we might see a certain behaviour. Hence computation will be performed with initial conditions chosen purely because they meet the required boundary conditions, and ran for a sufficient length of time.

Chapter 2

The Physics of Glaciers

The term glacier is well known, though most only know that it has something to do with ice, so what exactly is one? The official definition from the AMS glossary [14] says:

> *"`A glacier is a mass of land ice, formed by the further recrystallisation of firn, flowing continuously from higher to lower elevations."'*

So according to the definition, a glacier is essentially a river of ice flowing down a mountain, where the ice is made up from firn (German for old snow) recrystallising. Knowing how they form is important if we are interested in how they move. In order for glaciers to form they first need enough snow over the winter period to be able to survive through the summer, i.e. more accumulation of snow than is lost through melting and evaporation. This needs to be repeated over a number of succesive years, and as more snow

push it down the mountain. It is this feature which is of special interest in this dissertation.

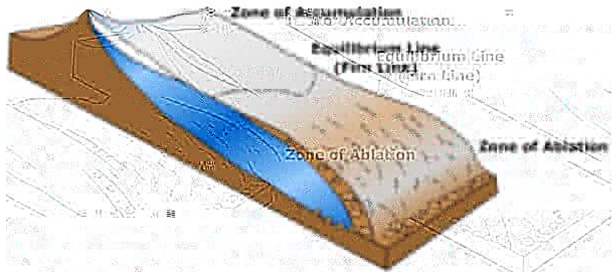

Figure 2.1: *Glacier Zones*

Each individual glacier tends to have reasonably steady flow, but between glaciers this flow can vary considerably. Some glaciers are even prone to surges, where they barely move for years before advancing very quickly; generally a few kilometres over a few months.

One of the nice things about glaciers is that they move similarly to a viscous fluid, though with a very high viscosity of around $10^{12} Pa{:}s$, and for comparison this is roughly 10^{15} times that of water [8]. However we cannot use viscous theory to measure flow, since glaciers are unique in experiencing something called *basal sliding*. As the ice is flowing down the mountain, friction is generated, melting the ice which makes contact with the surface, causing the base to slide. This can also be caused by geothermal heat below the surface. These factors can be used to set up a mathematical model.

Chapter 3

A One-Dimensional Model

Consider a glacier on a flat bed occupying the region $[0, b(t)]$ as shown in Fig.3.1. Let H be the thickness of the ice. At the ends of this domain we have two boundary conditions, $H = 0$ at $x = b(t)$ and $\frac{\partial H}{\partial x} = 0$ at $x = 0$.

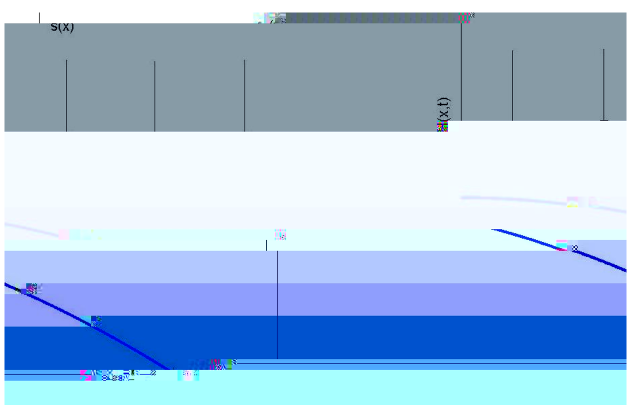

Figure 3.1: *One-Dimensional Domain*

As a starting point we are going to derive a simple model for glaciers. This takes the form

$@H(x; t$

From Van Der Veen[4] the driving stress is given by

$$\tau_{dx} = -\rho g H \frac{\partial h}{\partial x},$$

with ρ the ice density, g representing gravity, and h representing the ice thickness plus the surface elevation. On a flat bed there is no surface elevation so we may put $h = H$. Putting all of these terms together we get an equation for the vertical mean velocity

$$u = -\frac{2AH}{n+2}\rho^n g^n H^n \left(\frac{\partial H}{\partial x}\right)^n.$$

Most of the parameters in the model may be set as constant to simplify the model as much as possible, giving us

$$u = cH^{n+1}\left(\frac{\partial H}{\partial x}\right)^n.$$

From Roberts [1] we set $c = 0.000022765$. Putting the velocity back in to equation 3.2 we get the model equation

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left(cH^5 H_x^3\right) + s(x), \tag{3.3}$$

which incorporates non-linear diffusion and a source term.

3.2 Initial Conditions

Initially, $x \in [0, 1]$, (i.e. $b(0) = 1$), and the ice thickness is defined as

$$H = (1 - x^2)^\alpha, \tag{3.4}$$

where α here is set to 1 in the first instance, but will come into play later on. The form (3.4) is chosen since it satisfies the boundary conditions, but it is clearly not unique. The snow term will be approximated for all time by a linear function

$$s = e(1 - dx); \tag{3.5}$$

where d & e are the snow parameters, typically set to be 0.5 [1]. This results in a positive snow term up until $x = 2$, after which ablation takes over and the snow term becomes negative, making it harder for the glacier to survive further down the mountain.

An interesting physical concept to begin with is to look at what happens to the integral of the ice thickness over the whole domain (the volume) i.e.

$$\int_0^{b(t)} H(x;t)dx = \Phi(t); \text{ say.} \tag{3.6}$$

and see how this changes over time.
Using Leibniz's integral rule, and applying our boundary conditions we see that

$$\begin{aligned}
\frac{d}{dt}\int_0^{b(t)} H(x;t)dx &= \int_0^{b(t)} \frac{\partial H}{\partial t}dx + H(b(t);t)\frac{db(t)}{dt} - 0 \\
&= \int_0^{b(t)} \frac{\partial}{\partial x}\left(cH^5H_x^3\right)dx + \int_0^{b(t)} s(x)dx \\
&= \left[cH^5H_x^3\right]_0^{b(t)} + \int_0^{b(t)} s(x)dx \\
&= \int_0^{b(t)} s(x)dx.
\end{aligned} \tag{3.7}$$

The physical equivalent says that any change in the integral of ice thickness over the whole glacier, or equivalently any change in the ice volume, is due only to the snow term, which represents the net accumulation/ablation of snow over the whole glacier.

3.3 Velocity

In order to use a moving grid we need be able to define a velocity, v, at any arbitrary point. As with most of the variables this velocity is vertically averaged through the ice thickness. To define this velocity we need to make some form of assumption, of which there are two that will be considered here, then computationally analysed over the next few chapters.

The first method (in section 3.3.1) is to assume equation 3.7 in any moving subdomain $[0, \hat{b}(t)]$ of $[0, b(t)]$, holds with $\hat{b}(t)$ instead of $b(t)$. In physical terms this velocity is such that the ice volume changes only due to accumulation/ablation of snow locally. The second method (in section 3.3.2) assumes that the normalised volume $\frac{1}{\,}\int_0^{\hat{b}(t)} H(x, t)dx$ remains constant in time, i.e. the ice volume fraction remains constant as the glacier moves.

There are other assumptions that have been made in generating a velocity, and could be made here, such as assuming that each point in the domain is connected to its neighbours via 'springs', but they are not considered here.

3.3.1 Method 1 - Subdomain Assumption

In the first method assume that

$$\frac{d}{dt}\int_0^{\hat{b}(t)} H(x, t)dx = \int_0^{\hat{b}(t)} s(x)dx, \tag{3.8}$$

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3.3dt3.3dt3.3H

3.3.2 Method 2 - Normalisation Assumption

Method 2 assumes that an ice volume fraction remains constant in time, i.e.

$$\frac{d}{dt}\left(\frac{1}{(t)}\int_0^{\hat{x}(t)} H(x;t)dx\right) = 0 \tag{3.10}$$

$$\Rightarrow \frac{1}{(t)}\int_0^{\hat{x}(t)} H(x;t)dx = \;;\ \text{say}, \tag{3.11}$$

where we define (t) to be the total volume of the ice, equation (3.6).

Differentiating 3.10, we get

$$\frac{'}{2}\int_0^{\hat{x}(t)} H(x;t)dx + \frac{1}{}\frac{d}{dt}\int_0^{\hat{x}(t)} H(x;t)dx = 0;$$

and then by applying by Leibniz's integral rule and our boundary conditions

$$\frac{'}{2}\int_0^{\hat{x}(t)} H(x;t)dx + \frac{1}{}\quad D + \int_0^{\hat{x}} s dx + H(\hat{x};t)\frac{d\hat{x}}{dt} = 0; \tag{3.12}$$

where $' = \frac{d}{dt}$. From here we can then rearrange to get the velocity.

$$\begin{aligned}
\frac{d\hat{x}}{dt} &= \frac{'}{H}\int_0^{\hat{x}(t)} H(x;t)dx \quad \frac{D}{H} \quad \frac{1}{H}\int_0^{\hat{x}} s dx \\
&= \frac{'}{H} \quad cH^4H_x^3 \quad \frac{1}{H}\int_0^{\hat{x}} s dx \\
&= \frac{1}{H} \quad ' \quad \int_0^{\hat{x}} s dx \quad cH^4H_x^3
\end{aligned} \tag{3.13}$$

Note that we have substituted the constant from Equation 3.11, and that the final term in the equation is the same as the velocity in method one. Note also that when $\hat{x} = b(t)$, at the snout of the glacier, the first term disappears and the velocity reduces to the same as in method 1.

Equation (3.16) requires $'$ (= $\frac{d}{dt}$), from 3.7

$$
\begin{aligned}
' &= \frac{d}{dt}\int_0^b H\,dx \\
&= \int_0^b s\,dx. \qquad (3.14)
\end{aligned}
$$

The analysis of this method will be carried out in Chapter 5.

3.4 Snout Profile

From (3.9) we have the useful form

$$
\begin{aligned}
v = \frac{dx_b}{dt} &= c(H^{4/3}H_x)^3 \\
&= c\left(\frac{3}{7}(H^{7/3})_x\right)^3 \\
&= \frac{27}{343}c\left((H^{7/3})_x\right)^3. \qquad (3.15)
\end{aligned}
$$

Which shows that v does not have this problem at the right hand boundary, since it is perfectly possible for $(H^{7/3})_x$ to be non-zero, as long as H_x is infinite. Hence this is the velocity for method 1, and method 2 becomes

$$
v = \frac{1}{H}\left(' - \int_0^{x_b} s\,dx - \frac{27}{343}c\left((H^{7/3})_x\right)^3 \right). \qquad (3.16)
$$

This point will be examined further in Chapter 4.

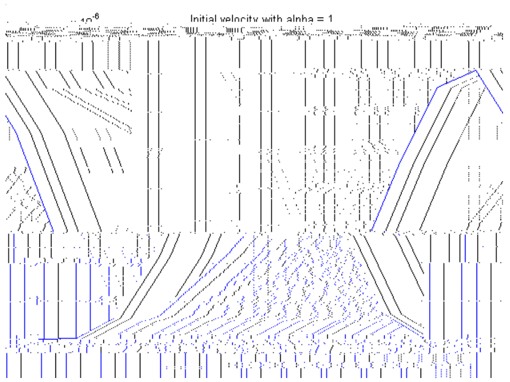
When expressing the velocity in this manner it is worth considering what will happen when we substitute the initial expression for H, from equation (3.4).

$$\begin{aligned} H^{\frac{7}{3}} &= (1 - x^2)^{\frac{7}{3}\lambda} \\ (H^{7/3})_x &= -2x.\frac{7}{3}\lambda(1 - x^2)^{\frac{7}{3}\lambda - 1}. \end{aligned} \tag{3.17}$$

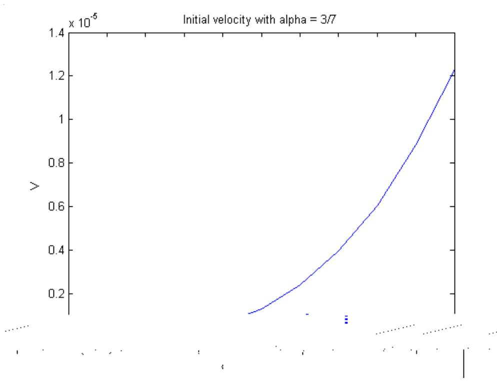
This has some interesting properties as $x \to 1$, depending on the value of λ.

$$\begin{aligned} \text{Case 1: } \frac{7}{3}\lambda &> 1; \Rightarrow (H^{7/3})_x \text{ is zero} \\ \text{Case 2: } \frac{7}{3}\lambda &< 1; \Rightarrow (H^{7/3})_x \text{ is infinite} \\ \text{Case 3: } \frac{7}{3}\lambda &= 1; \Rightarrow (H^{7/3})_x \text{ is finite} \end{aligned}$$

In case 1, from equation (3.15), the initial velocity of the snout of the glacier is zero, and it is stationary; the chosen setting of $\lambda = 1$ satisfies this, as shown in Fig.3.2(a). In case 2, we get an infinite velocity, which is not physical and the model is incorrect. In case 3 we get a finite initial velocity value at the snout when $\lambda = \frac{3}{7}$, which would be the point when the right hand boundary starts to move, as observed in Fig.3.2(b)



(a) Initial V under case 1 (α = 1)



(b) Initial V under case 3 (α = 3/7)

(c) Initial Ice Thickness under di erent α

Figure 3.2: *Analysis of initial velocity*

3.4.1 Surface Elevation

A at bed model can be limiting when considering physical examples. For glaciers that are situated in the mountains the bed will almost certainly be sloped. Consider now the domain in Fig.3.3, where we have a linear slope. H

To calculate the grid velocity we are going to use the same assumption that is made in method 1 (section 3.3.1). Following through we end up with a similar equation

$$(v =)\frac{ds(t)}{dt} = \frac{D(H; H_x)}{H} = cH^4 h_x^3. \tag{3.19}$$

For the velocity to be finite at the snout where $H = 0$, we must have $h_x = 1$. Since we cannot express equation (3.19) in the same form that we did before the problem at the right hand boundary still occurs.

In addition we also need to define the slope, . This will be set to

$$= x + 5; \tag{3.20}$$

chosen so that the glacier is initially small in comparison with the slope.

We now use these model descriptions to analyse and compute glacier behaviour.

Chapter 4

Computation from Method 1: Subdomain Assumption

4.1 Numerical Grid

Equation 3.1 is non-linear and thus di cult to solve analytically, so we seek a numerical approximation via a grid. As a 1D problem, the domain only needs to be divided up along the x-direction. Since the problem involves a moving boundary, a natural description is to use a moving grid. The grid points will need to be updated every time step, since as the glacier moves we expect the grid to follow, giving a moving grid problem.

The initial grid is chosen to be evenly spaced at the initial time, although there is potential to introduce a non-evenly spaced grid, particularly near the right hand boundary to give more information about the velocity and movement at the snout of the glacier.

4.2 Numerical Approximation

For the velocity in equation (3.15) we can use our grid from above to form an approximation using an upwind difference

$$v_i = \left(\frac{27}{343} c \, \frac{H_i^{7/3} - H_{i-1}^{7/3}}{x_i - x_{i-1}} \right)^3 . \tag{4.1}$$

At each time step the velocity can be calculated, from which we can then use equation (4.1) to update the x and H values. For the new x values we approximate Equation 3.15 using forward Euler time stepping,

$$\begin{aligned} \frac{d\hat{x}(t)}{dt} &= v \\ \frac{\hat{x}^{n+1} - \hat{x}^{n}}{\Delta t} &= v \\ \hat{x}^{n+1} &= \hat{x}^{n} + v . \Delta t . \end{aligned} \tag{4.2}$$

To determine H we go back to the assumption we made in equation (3.8), and use the same time-stepping scheme. Note that the limits in (3.8) have been changed to allow the midpoint rule to be applied for computational simplicity.

$$\frac{\int_{j-1}^{j+1} H^{n+1} dx - \int_{j-1}^{j+1} H^{n} dx}{\Delta t} = \int_{j-1}^{j+1} s dx .$$

Then, using the midpoint rule we get

$$(x_{j+1}^{n+1} - x_{j-1}^{n+1}) H_j^{n+1} - (x_{j+1}^{n} - x_{j-1}^{n}) H_j^{n} = \Delta t (x_{j+1}^{n} - x_{j-1}^{n}) s_j^{n} ;$$

giving

$$H_j^{n+1} = \frac{(x_{j+1}^{n} - x_{j-1}^{n})}{(x_{j+1}^{n+1} - x_{j-1}^{n+1})} (H_j^{n} + \Delta t s_j^{n}) . \tag{4.3}$$

4.3 Results

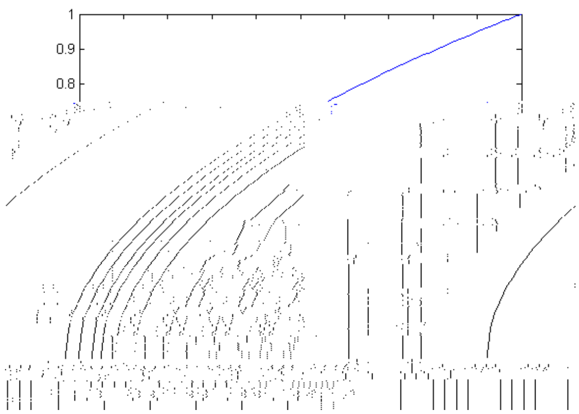
What does the velocity do over time under the assumption in equation (3.8)? From Fig.4.1 we can see that the velocity builds up into a dome shape before the peak begins to move towards the right hand boundary, eventually reaching it and pushing the boundary of the glacier into movement. This resembles the solution of a non-linear di erential equation that generates a shock, something which we consider further in Section 7.

How does the ice thickness behave under this velocity pro le? From Fig.4.2(a), we can see that the glacier does not move past its initial endpoint until t = 10000, after the time that it has been reached by the shock. The change in ice thickness appears to be initially dominated by the build up of snow rather than the di usion term, hence why the glacier appears to grow high before some of the snow term becomes negative and the di usion can take over, pulling the glacier back down to a more reasonable shape. From a physical perspective this perhaps is not the most realistic of movements.

not possible numerically, due to the grid spacing.

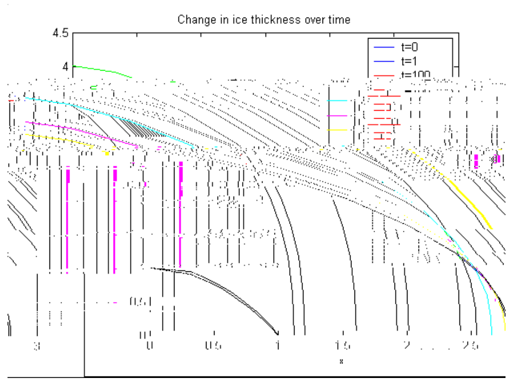
4.4 Non-Evenly Spaced Grid

Since it is clear that the important area is the snout perhaps it might be an idea to focus the grid on this point to improve accuracy there. To do this we declare a new space variable, , which takes on the role of the existing evenly spaced grid. Then we de ne $x = p$—. As you can see in Fig. 4.3(a), the region close to the snout at 1 is covered by more points under x. The main question is does this have an impact on the other variables. Looking at Fig.4.3, it would appear that the general shape is very similar, though the lack of information near the left hand boundary does cause a bit of inaccuracy, which is to be expected since this area is less well resolved. Perhaps the most interesting fact is that it appears that the increased resolution of the non-even grid, Fig.4.3(c) starts moving before the evenly spaced grid, judging by the location of the line $t = 100$. This might lead to an underestimated waiting time if calculated on the evenly spaced grid.



(a) Difference in grid spacing variables

(b) Even spaced grid



(c) Non-Even grid

Figure 4.3: *Computation using the non-evenly spaced grid*

Chapter 5

Computation from Method 2: Normalisation Assumption

Calculating the velocity by method 2 from section 3.3.2 requires a little more work than for the subdomain assumption. Recalling equation 3.16, we now have more terms to handle. To start with, the constant , which will be a vector, can be de ned at the outset since this will not need to be updated. Any integrals can be estimated numerically by the trapezium rule via a sum, rstly approximating by

$$= \sum_{j=0} 0.5(H_j + H_{j \quad 1})(x_j \quad x_j$$

j3]TJ/F2i 11.9552 Tf 7.039 1.792821 11.327 (y)]

To update H, we note that since is constant in time, then it follows that at any time n

$$\frac{1}{(t_n)}\int_0^{b^n} H^n dx = \frac{1}{(0)}\int_0^{b^0} H^0 dx$$

$$\frac{1}{(t_n)}\int_{x_i^n\;1}^{x_{i+1}^n} H^n dx = \frac{1}{(0)}\int_{x_i^0\;1}^{x_{i+1}^0} H^0 dx$$

$$\frac{1}{(t_n)} H_i^n (x_{i+1}^n \quad x_i^n\;1) = \frac{1}{(0)} H_i^n (x_{i+1}^0 \quad x_i^0\;1)$$

$$H^n$$

Chapter 6

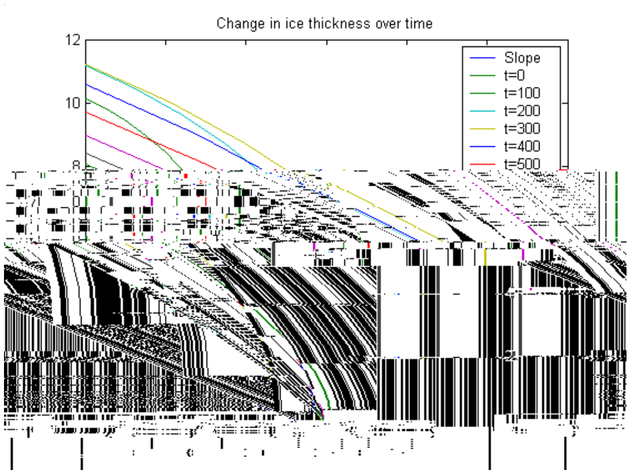
Surface Elevation Model

For the most part the theory for the surface elevation model is similar to Chapter 4, as we are making the same assumption for a different domain. The slope is defined as in equation (3.20), the ice thickness H will be the same as before, and h is defined as

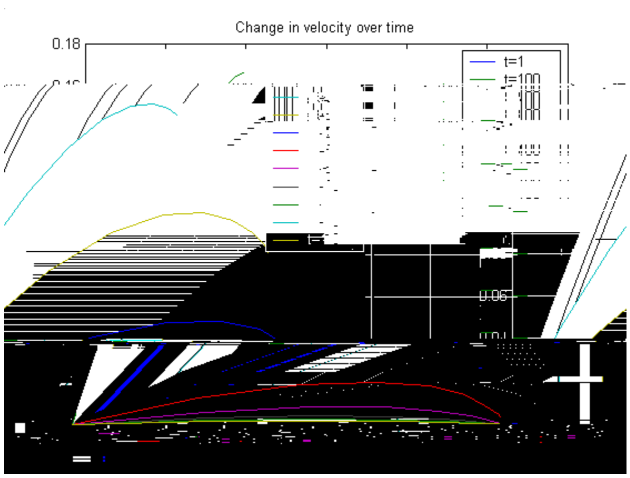
$$h = H + \quad :$$

As we would expect, the slope increases the velocity and causes the glacier to move quicker down the mountain, something which is clearly evident in Fig.6.1(a). However once the glacier is off the slope it seems to grind to a halt, which we see in the velocity plot, Fig.6.1(b). This is due to the linear snow term, since once x gets past the value 2 we see a negative contribution of snow (ablation), which becomes increasingly negative the further down we go.

Once the glacier has left the slope, the gradient decreases, so the diffusion term has less effect and from Chapter 4 we saw that the snow term is dominant under the subdomain assumptions.



(a) Ice Thickness



(b) Velocity

Figure 6.1: *Results with surface elevation*

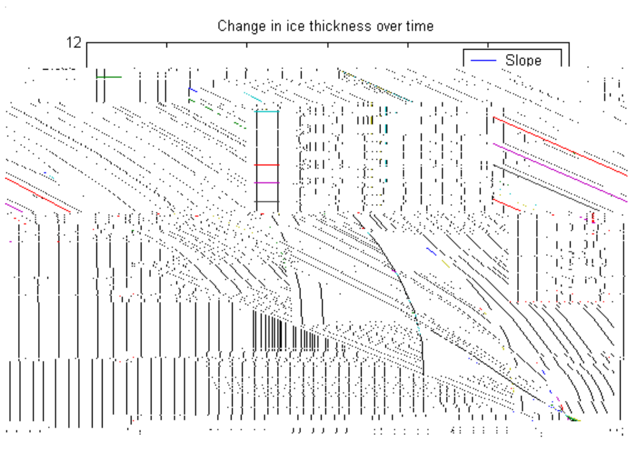
6.1 Basal Sliding

At the end of chapter 2 we mentioned the concept of basal sliding. Depending on the characteristics of a glacier, this can account for a significant part of its movement. Since the models we have been using involve a vertically averaged velocity, the concept of basal sliding need only be considered as an additional velocity rather than a velocity at the base. Van Der Veen [4] proposed including the basal sliding as an additional part of the diffusion term. From 3.1

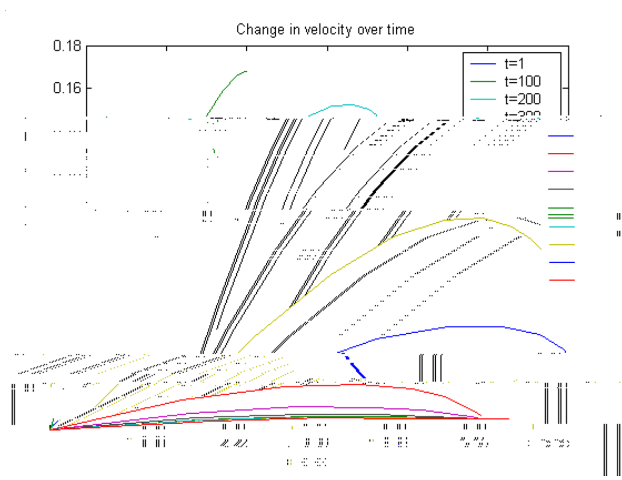
$$\frac{\partial H(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(cH(x,t)^5 H_x(x,t)^3 + H(x,t)V_{bs}(x,t)\right) + s(x) \tag{6.1}$$

where V_{bs}

never slows down to zero, and if left long enough the glacier would keep moving and stretching out until it becomes a thin sheet.



(a) Ice Thickness



(b) Velocity

Figure 6.2: *Results with surface elevation and Basal Sliding*

Chapter 7

A Shock Equation

From equation 3.9 recall that under the subdomain assumption, the velocity is given by

These can be substituted into 7.2 to give

$$\begin{aligned} v_t &= \quad 3cH^4H_x^2(s_x \quad H_{xx}v \quad 2H_xv_x \quad Hv_{xx}) \quad 4cH^3H_x^3(s \quad H_xv \quad Hv_x) \\ &= 3cH^5H_x^2v_{xx} + (10cH^4H_x^3)v_x + (3cH^4H_x^2H_xx + 4cH^3H_x^4)v \quad 3cH^4H_x^2s_x \quad 4cH^3H_x^3s \\ &= 3cH^5H_x^2v_{xx} \quad 10vv_x \quad v_xv \quad 3cH^4H_x^2s_x \quad 4cH^3H_x^3s; \end{aligned}$$

and nally rearranged in the form of a Burgers-like equation with extra source terms.

$$v_t + 11vv_x = 3cH^5H_x^2v_{xx} \quad 3cH^4H_x^2s_x \quad 4cH^3H_x^3s: \tag{7.5}$$

We shall use this equation to characterise the evolution of v and estimate a waiting time.

7.1 Numerical Approximation to a Burgers equation

With equation 7.5 we can form an approximation to the change in velocity over time. We expect this to re ect the change of velocity that we see in Fig. 4.1.

Each of the velocity derivative terms have been numerically approximated, again using an upwind di erence,

$$v_t = \frac{v_i^{n+1} \quad v_i^n}{t} \tag{7.6}$$

$$vv_x = \frac{1}{2}\frac{(v_i^n)^2 \quad (v_{i\ 1}^n)^2}{x} \tag{7.7}$$

$$v_{xx} = \frac{v_{i+1}^n \quad 2v_i^n + v_{i\ 1}^n}{(\ x)^2}: \tag{7.8}$$

The snow derivative term also appears in equation (7.5), but since s is a linear function (equation (3.5)), the derivative is constant, in our case

$$s_x = \quad 0{:}25{:}$$

Substituting all of this into equation 7.5,

$$\frac{v_i^{n+1} \quad v_i^n}{t} + \frac{11}{2}\frac{(v_i^n)^2 \quad (v_{i\ 1}^n)^2}{x} = 3cH^5H_x^2\frac{v_{i+1}^n \quad 2v_i^n + v_{i\ 1}^n}{(\quad x)^2} + \frac{3}{4}cH^4H_x^2 \quad 4cH^3H_x^3 s; \tag{7.9}$$

which can be rearranged to give an explicit equation for v_i^{n+1}.

In order to implement 7.9 we need an initial state for velocity, which we can approximate from 4.1, by

$$v_i^0 = \quad \frac{27}{343}c \quad \frac{H_i^{7=3} \quad H_{i\ 1}^{7=3}}{x_i \quad x_{i\ 1}} \quad ^3 : \tag{7.10}$$

Note that since equation (7.5) depends on H (and as a result x), these variables will also need to be updated every time loop, which can be done using the same methods as we used in Chapter 4.

Plotting the solution to this equation, Fig. 7.1, it is encouraging to see that the velocity changes in a similar manner to what we saw in Fig. 4.1.

7.2 Characteristics

Using equation (7.5) as a check that our method produces the correct results is useful, but we can also use the equation to estimate when the shock occurs. To do this we use characteristics theory to observe that by the chain rule

$$v_t + \frac{dx}{dt}v_x = \frac{dv}{dt}{:}$$

Under this condition we see that

$$v = v_0; \text{ say,} \tag{7.16}$$

which, when substituted into equation (7.14) implies that

$$x = v_0 t + x_0. \tag{7.17}$$

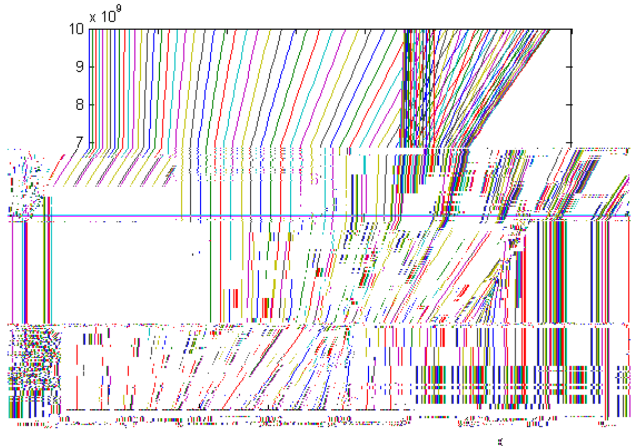
This gives us the set of characteristics which we plot in Fig.7.3(a), and we can see that the characteristics cross towards the right. In a characteristics plot for a conservation law of the form of equation (7.13), any time lines cross a shock is generated, which moves forward in time. It is interesting to note that the shock occurs where the gradient of initial velocity was steepest, (see Fig.3.2(a)

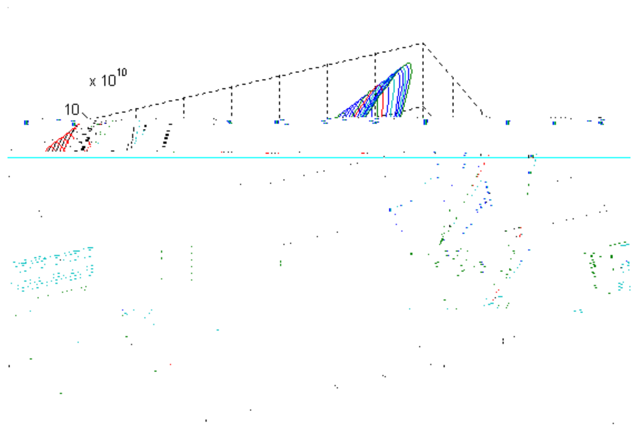
The envelope forms, and the shock appears at the earliest possible time satisfying (7.20), i.e.

$$t_{m}in = \frac{1}{max[\;\;v_0^{\prime}(x_0)]}: \qquad (7.21)$$

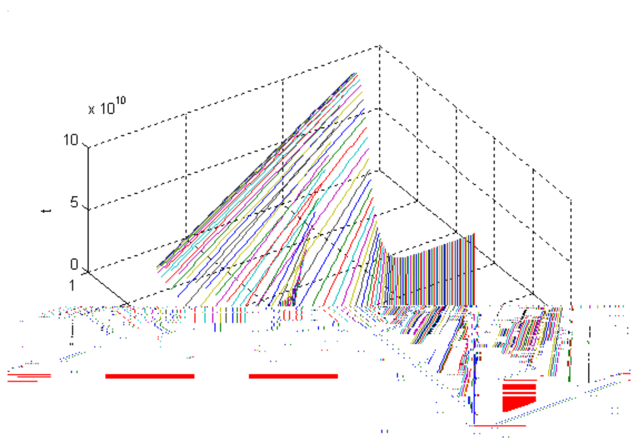
Now we have a time that the shock will occur in the homogeneous case, but we are also interested in the shock speed since this will help us predict the time the shock arrives at the boundary.

One way of calculating the shock speed is to look at the conservation property (Whitham, 1974). This work was used by C.P. Reeves, and states that we can replace the overturned part of the curve by a vertical line such that the areas A





(a) Progression of shock through time



(b) Progression of shock points through time

Figure 7.4: *3D plots of Characteristics*

case. Due to the scaling the results here are speculative, and their value needs to be investigated.

Chapter 8

Conclusions and Further Work

8.1 Summary

This dissertation has looked at a number of techniques for modelling a one-dimensional glacier model using a moving grid. In addition we observed that glaciers experienced a waiting time and required certain circumstances before they began to move. As such we looked at combining the work of Roberts [1] on the 1D model, and Stojsavljevic [10] on waiting times for PDEs.

Firstly we took the flat bed model and used two methods of extracting the grid velocity. The first was a simpler solution, assuming a subdomain held the same properties as the whole domain. This was a physical approach,

numerically and we saw similar results, which acted as a reinforcement on the work we had already done. Then we applied the method of characteristics to get a system of ODEs that we solved using Matlab's inbuilt solver. Firstly the homogeneous case gave good results that we were able to use to get an explicit formula for the shock formula. The inhomogeneous problem gave some output, but only after scaling down the ice thickness to avoid blow up. We noticed that the velocity increases while overturning, which was not seen in the homogeneous case. These results are dubious.

8.2 Further Work

Throughout this dissertation there were many2Rsconsiderertati1(this)-4asion whic and in chapter 6 we saw that the snow term ground the ice to a halt as soon as it left the slope. This is something which should be addressed before any

Figure 8.1: *Effect of Abrasion on Basal Sliding*

sides of the glacier, and the type would depend on if the sides meet a wall (no flux condition) or if they just curve to the ground, ($H = 0$). The model itself will take the form

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