VARIATIONAL APPROACH IN WEIGHTED SOBOLEV SPACES TO SCATTERING BY UNBOUNDED ROUGH SURFACES

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Abstract. We consider the problem of scattering of time harmonic acoustic waves by an unbounded sound soft surface which is assumed to lie within a finite distance of some plane. The paper is concerned with the study of an equivalent variational formulation of this problem set in a scale of weighted Sobolev spaces. We prove well-posedness of this variational formulation in an energy space with weights which extends previous results in the unweighted setting (Chandler-Wilde & Monk, SIAM J Math Anal 37 (2005), 598-618) to more general inhomogeneous terms in the Helmholtz equation. In particular, in the two-dimensional case, our approach covers the problem of plane wave incidence, whereas in the 3D case incident spherical and cylindrical waves can be treated. As a further application of our results we analyse a finite section type approximation, whereby the variational problem posed on an infinite layer is approximated by a variational problem on a bounded region.

1. Introduction. This paper is concerned with the analysis of problems of scattering by unbounded surfaces, in particular with what are termed *rough surface scattering problems* in the engineering literature. By the phrase *rough surface*, we will denote throughout a surface which is a (usually non-local) perturbation of an infinite plane surface such that the surface lies within a finite distance of the original plane. Rough surface scattering problems in this sense arise frequently in applications, for example in modeling acoustic and electromagnetic wave propagation over outdoor ground and sea surfaces, and have been studied extensively in the physics and engineering literature from the points of view of developing effective numerical algorithms or asymptotic or statistical approximation methods (see e.g. Ogilvy 30], Voronovich 39], Saillard & Sentenac 32], Warnick & Chew 40], DeSanto 18], and Elfouhaily and Guerin 19]).

Despite this extensive practical interest, relatively little mathematical analysis of these problems has been carried out. In particular, only in the last four years have the first results been obtained establishing well-posedness for three-dimensional rough surface scattering problems, using integral equation methods (see Chandler-Wilde, Heinemeyer & Potthast 13, 14], Thomas 36]) or variational formulations (see Chandler-Wilde, Monk & Thomas 11, 15], Thomas 36]). The variational approach proposed in 11] for the sound soft acoustic problem leads to explicit bounds on the solution in terms of the data and applies to a rather general class of non-smooth unbounded surfaces. The approach in 11] is extended to more general acoustic scattering problems in 36], including problems of scattering by impedance surfaces and by inhomogeneous layers (and see 15]).

In contrast to the general case of a non-locally perturbed plane surface, there is already a vast literature on the variational approach applied to periodic diffractive structures (diffraction gratings) or to locally perturbed plane scatterers; see, e.g., Kirsch 25], Bonnet-Bendhia & Starling 6], Elschner & Schmidt 20], Bao & Dobson 5], Elschner, Hinder, Penzel & Schmidt 21], Ammari, Bao & Wood 1], and Elschner & Yamamoto 22]. The assumption made in all of these papers leads to a variational problem over a bounded region, so that compact imbedding arguments can be applied and the sesquilinear form that arises satisfies a Gårding inequality which simplifies the mathematical arguments considerably compared to the cases studied in 11], 15] and 36].

In this paper we will rigorously analyze time harmonic acoustic scattering, seeking to solve the Helmholtz equation with wave number $k >$,

$$\Delta u + k^2 u = g,$$

in the perturbed half-plane or half-space $D \subset \mathbb{R}^n$, $n =$, . The scattering surface $= \partial D$ is assumed to lie within a finite distance of some plane; for example it may be the graph of an arbitrary bounded continuous

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function. While the methods we use and results we derive can be adapted to other boundary conditions, to keep things specific and to make use of earlier results 11, 15], we will restrict our attention to the simplest case when a homogeneous Dirichlet boundary condition $\mathbf{u} =$ holds on . The problem formulation is completed by a suitable radiation condition, expressing that the wave scattered by the surface must radiate away from the surface.

This paper is closest in its results to Chandler-Wilde & Monk 11], who studied the same Dirichlet scattering problem. Following 11], we introduce an equivalent variational formulation of this problem set in an infinite layer $\mathbf{S}_0$ of finite thickness between the surface and some plane $_0$ lying above that surface on which the solution is required to satisfy a non-local boundary condition involving the exact Dirichlet to Neumann map $\mathbf{T}$. This condition is often used in a formal manner in the rough surface scattering literature (e.g. 18]), that, above the rough surface and the support of $\mathbf{g}$, the solution can be represented in integral form as a superposition of upward traveling and evanescent plane waves. This radiation condition is equivalent to the upward propagating radiation condition proposed for two-dimensional rough surface scattering problems in 10], and has recently been analyzed carefully in the 2D case by Arens and Hohage 4]. Arens and Hohage also propose a further equivalent radiation condition (a 'pole condition').

In Sections 2 and 3 we formulate the boundary value problem and its variational formulation precisely, and give the details about our assumptions on $\mathbf{D}$ and about the radiation condition we impose. Section 3 is also devoted to new continuity properties of the DtN map $\mathbf{T}$ in weighted Sobolev spaces on $_0$.

In Section 4 we study the well-posedness of the variational formulation in an energy space with weights which decay or increase polynomially as a function of radial distance within the layer $\mathbf{S}_0$. Our main result, Theorem 4.1, is to show, for a range of increasing and decreasing weights, that the problem is well-posed in the weighted space setting if and only if it is well-posed in the unweighted space setting. This result depends on technical estimates of the commutator of the DtN map $\mathbf{T}$ and the operation of multiplication by the weight function; see Theorem 3.1. Combining this result with previous results on well-posedness in the unweighted setting for sound soft scattering 11], we are able to show well-posedness in a weighted space setting.

In Section 5, to illustrate the importance of these results, we make two applications. First, in the two-dimensional case, we prove existence of solution to the problem of plane wave scattering by an unbounded sound soft where $\mathbf{e_n}$ denotes the unit vector in direction $\mathbf{x_n}$. Condition (2.2) is satisfied if is the graph of a continuous function, but also allows more general domains.

We now introduce weighted $\mathbf{L}^2$ and Sobolev spaces. For $\varrho \in \mathbb{R}$, $\mathbf{l} \in \mathbb{N}$ and a domain $\mathbf{G} \subset \mathbb{R}^{\mathbf{n}}$, define the Hilbert spaces

$$\mathbf{L}^2_\varrho \; \mathbf{G} \;=\; +\mathbf{x}^2 \;{}^{-\varrho/2}\mathbf{L}^2\; \mathbf{G}\;, \quad \mathbf{H}^{\mathbf{l}}_\varrho\; \mathbf{G} \;=\; +\mathbf{x}^2\;{}^{-\varrho/2}\mathbf{H}^{\mathbf{l}}\;\mathbf{G}\;,$$

equipped with the corresponding canonical norm and scalar product. The space $\mathbf{V_{h,\varrho}}$ is then defined, for $\mathbf{h} \geq$, as the closure of $\{\mathbf{u}|_{\mathbf{S}_h} \;\; \mathbf{u} \in \mathbf{C}_0^\infty \;\; \mathbf{D}\;\}$ in the norm

$$\|\mathbf{u}\|_{\mathbf{V}_{h,\varrho}} = \|\mathbf{u}\|_{\mathbf{H}^1_\varrho(\mathbf{S}_h)} = \left(\int_{\mathbf{S}_h}\left(\Big| \quad +\mathbf{x}^2\;{}^{\varrho/2}\mathbf{u}\Big|^2 + \Big|\nabla \quad +\mathbf{x}^2\;{}^{\varrho/2}\mathbf{u}\;\Big|^2\right)\mathbf{d}\;\right)^{1/2}. \tag{2.3}$$

We set $\mathbf{V}_{0,\varrho} = \mathbf{V}_\varrho$ in the following, which is the energy space for our variational problem. Moreover, we introduce

$$\mathbf{H}^{\mathbf{s}}_\varrho \quad \mathbf{h} \;=\; +\mathbf{x}^2\;{}^{-\varrho/2}\mathbf{H}^{\mathbf{s}} \quad \mathbf{h}\;, \quad \mathbf{s}, \varrho \in \mathbb{R}\,,$$

where $\mathbf{H}^{\mathbf{s}} \quad \mathbf{h}$ is identified

Remark 2.2. *We note (and this is important in our later applications) that there is a degree of arbitrariness in our radiation conditions* (2.4) *and* (2.5). *By this we mean that one could replace* x_n *in* (2.4) *by* $x_n - c$, *for any* $c >$ *(in fact for any* $c \in \mathbb{R}$ *such that* pp $g \subset S_c$ *and* $U_c \subset D$*); the corresponding change to* (2.5) *would be to replace*

3. The Dirichlet to Neumann Map and Variational Formulation. We now consider a variational formulation in weighted Sobolev spaces of the above boundary value problem, which involves the Dirichlet-to-Neumann operator on the artifical boundary $_0$. As in 11] for $=$, there exist continuous trace operators

$$\mathbf{y}_- \quad \mathbf{V} \rightarrow \mathbf{H}^{1/2} \quad _0 \mathbf{,} \ \mathbf{y}_+ \quad \mathbf{H}^1 \ \mathbf{U}_0 \backslash \mathbf{U_h} \ \rightarrow \mathbf{H}^{1/2} \quad _0 \mathbf{,} \ \mathbf{h} > \quad \mathbf{.}$$

Moreover, if $\mathbf{u}_0 \in \mathbf{C}_0^{\infty} \quad _0$ and $\mathbf{u}$ is given by giv

Note that this sesquilinear form is well-defined and continuous on $\mathbf{V}_{\square} \times \mathbf{V}_{-\square}$ for $|\square| <$ as a consequence of Lemma 3.3 with $\mathbf{s} = \ /$.

The variational formulation (V). Given $\mathbf{g} \in \mathbf{L}^2_{\square}\ \mathbf{S}_0$, $|\square| <$, find $\mathbf{u} \in \mathbf{V}_{\square}$ such that

$$\mathbf{B}\ \mathbf{u},\mathbf{v}\ = -\ \mathbf{g},\mathbf{v}\ , \quad \forall \mathbf{v} \in \mathbf{V}_{-\square}\,. \tag{3.4}$$

As in 11], the equivalence of (BVP) and (V) follows from the following weighted version of Lemma 3.2 in that paper.

LEMMA .4. *Let* $|\square| <$.

(i) If (2.4) *holds with* $\mathbf{u}_0 \in \mathbf{H}^{1/2}_{\square}\ \ _0$, *then* $\mathbf{u} \in \mathbf{H}^1_{\square}\ \mathbf{U}_0 \backslash$

Proof for $\varrho \neq$. Introduce equivalent norms $\|\mathbf{u}\|_{L^2_\varrho} = \| \; a^2 + x^2 \;\; ^{\varrho/2} u\|_{L^2}$ with parameter $a >$ and modify the norm (2.3) in V_ϱ correspondingly. We will choose $a >$ sufficiently large, and set, for $u \in V_\varrho$, $\phi \in V_{-\varrho}$,

$$v = \;\; a^2 + x^2 \;\; ^{\varrho/2} u \in V_0 , \quad \psi = \;\; a^2 + x^2 \;\; ^{-\varrho/2} \phi \in V_0 .$$

Then we obtain from (3.3)

$$B \;\; u, \phi \;\; = B \;\; v, \psi \;\; + K \;\; v, \psi \;\; , \tag{4.1}$$

where $K = K_1 + K_2$ with

$$\begin{aligned} K_1 \;\; v, \psi \;\; &= \;\; \nabla \;\; a^2 + x^2 \;\; ^{-\varrho/2} v, \nabla \;\; a^2 + x^2 \;\; ^{\varrho/2} \psi \;\; - \;\; \nabla v, \nabla \psi \\ &= \;\; v \nabla \;\; a^2 + x^2 \;\; ^{-\varrho/2}, \psi \nabla \;\; a^2 + x^2 \;\; ^{\varrho/2} + \;\; \nabla v, \psi \;\; a^2 + x^2 \;\; ^{-\varrho/2} \nabla \;\; a^2 + x^2 \;\; ^{\varrho/2} \\ &+ \;\; v \;\; a^2 + x^2 \;\; ^{\varrho/2} \nabla \;\; a^2 + x^2 \;\; ^{-\varrho/2}, \nabla \psi \end{aligned} \tag{4.2}$$

and

$$K_2 \;\; v, \psi \;\; = \int_{\Gamma_0} \Big\{ \;\; a^2 + x^2 \;\; ^{\varrho/2} \psi \, T \;\; a^2 + x^2$$

and since (cf. [11])

$$\|\mathbf{y}_{-}\mathbf{v}\|_{L^2(\Gamma_0)} \leq \mathbf{k}^{-1/2}|\mathbf{y}_{-}\mathbf{v}|_{H^{1/2}(\Gamma_0)} \leq \mathbf{k}^{-1/2}|\mathbf{v}|_{V_0},$$

(4.5) *implies that*

$$|\mathbf{K}_2\ \mathbf{v}, \psi\ | \leq \frac{\mathbf{c}\ \square}{\sqrt{\mathbf{ka}}}|\mathbf{v}|_{V_0}|\psi|_{V_0}.$$

Thus we have, for $\mathbf{ka} \geq$ *and* $|\square| <$ *,*

$$|\mathbf{K}_0\ \mathbf{v}, \psi\ | \leq \left(\frac{|\square|}{\mathbf{ka}}\left(\ + \frac{|\square|}{\mathbf{ka}}\right) + \frac{\mathbf{c}\ \square}{\sqrt{\mathbf{ka}}}\right)|\mathbf{v}|_{V_0}|\psi|_{V_0} \leq \frac{|\square| + \mathbf{c}\ \square}{\sqrt{\mathbf{ka}}}|\mathbf{v}|_{V_0}|\psi|_{V_0},$$

so that $\|\mathcal{K}_0\| \leq \ |\square| + \mathbf{c}\ \square\ \ /\sqrt{\mathbf{ka}}$*. Taking the bound*

$$\|\mathcal{B}_0^{-1}\| \leq \mathbf{y} = \ + \sqrt{\ }\ \kappa\ \kappa + \ ^2$$

from [11, Thm. 4.1] and using (4.6)*, we obtain the norm estimate*

$$\|\mathcal{B}_\square^{-1}\| \leq \ \mathbf{y}, \tag{4.7}$$

provided that

$$\|\mathcal{K}_0\| \leq \ |\square| + \mathbf{c}\ \square\ \ /\sqrt{\mathbf{ka}} \leq \frac{}{\mathbf{y}} \leq -\|\mathcal{B}_0^{-1}\|,$$

which holds for $\mathbf{a} \geq \ \mathbf{y}^2\ |\square| + \mathbf{c}\ \square\ \ ^2/\mathbf{k}$*. Since (V) written in operator form is the equation* $\mathcal{B}_\square\mathbf{u} = \mathbf{g}$*, where* $\mathbf{g} \in V_{-\square}^{\square}$ *is defined by* $\mathbf{g}\ \mathbf{v}\ = \ \mathbf{g}, \mathbf{v}$ *,* $\mathbf{v} \in V_{-\square}$*, this implies that the solution* $\mathbf{u}$ *of (V) satisfies*

$$|\mathbf{u}|_{V_\varrho} \leq \ \mathbf{y}|\mathbf{g}|_{V^*_{-\varrho}} \leq \ \mathbf{y}\mathbf{k}^{-1}|\mathbf{g}|_{L^2_\varrho(S_0)}, \tag{4.8}$$

provided $\mathbf{ka} \geq \qquad , \ \mathbf{y}^2\ |\square| + \mathbf{c}\ \square\ \ ^2$ *.*

5. Applications.

5.1. Plane Wave Incidence, Diffraction Gratings, and Other Scattering Problems. As an application of Theorem 4.1, the problem of plane wave incidence in the 2D case ($\mathbf{n} =$) can be treated. That is, it can be shown, in appropriate function spaces, that the scattering problem for plane wave incidence has exactly one solution in 2D (for a brief discussion of what goes wrong in the 3D case, see Remark 5.5 below, and see Remark 5.6 for details of 3D scattering problems which can be tackled by Theorem 4.1). The incident plane wave has the form

$$\mathbf{v}^{in} \quad = \ \mathrm{p}\ \mathbf{ik} \quad \theta\,\mathbf{x}_1 - \quad \theta\,\mathbf{x}_2\ \ ,$$

where θ is the angle of incidence, with $|\theta| < \pi/$. In this problem we look for the total field $\mathbf{v} = \mathbf{v}^{sc} + \mathbf{v}^{in}$, $\mathbf{v}^{sc}$ being the unknown scattered field, such that

$$\blacktriangle + \mathbf{k}^2\ \mathbf{v} = \ \text{ in } \mathbf{D}, \quad \mathbf{v} = \ \text{ on } \ , \tag{5.1}$$

and $\mathbf{v}^{sc}$ satisfies an appropriate radiation condition.

This 2D rough surface scattering problem with plane wave incidence has been treated before, by integral equation methods, in 17] where it is shown that there exists exactly one solution $\mathbf{v} \in \mathbf{C}^2\ \mathbf{D}\ \cap \mathbf{C}\ \mathbf{D}$ such that $\mathbf{v}$ is bounded in $\mathbf{S_h}$, for every $\mathbf{h} >$, and $\mathbf{v}^{sc}$ satisfies the radiation condition in the form (2.5) (termed the upwards propagating radiation condition (UPRC) in 17]). However, the proof in 17] is only for the case where $\partial\mathbf{D}$ is the graph of a sufficiently smooth ($\mathbf{C}^{1,1}$) function (this, or at least a restriction to Lyapunov surfaces, is an essential restriction due to the compactness arguments in the exist28010Ten this section we

and such that $v^{sc} = v - v^{in}$ satisfies the *ayleigh expansion radiation condition*, that

$$v^{sc} = \sum_{m \in Z} u_m \quad p \; ik \; \alpha_m x_1 + \beta_m x_2 \;, \qquad \in U_0, \tag{5.3}$$

where the u_m are complex constants, $\alpha_m = \theta + \pi m / kA$, and

$$\beta_m = \begin{cases} \sqrt{\;-\alpha_m^2}, & |\alpha_m| \leq \;, \\ i\sqrt{\alpha_m^2 - \;}, & |\alpha_m| > \;. \end{cases}$$

It is shown in 22] that (DGPW) has exactly one solution in the case that ∂D is the graph of an (A-periodic) Lipschitz function, by extending well-known arguments (see e.g. 25]), which apply in the case when ∂D is the graph of a smooth function, to the non-smooth Lipschitz case. The following corollary of Theorem 5.1 extends that result further to the much more general case where ∂D is only required to satisfy (2.1), (2.2), and (5.2).

Corollary 5.2. *Suppose that* (5.2) *holds. Then (DGPW) has exactly one solution, and this is the unique solution of (PW).*

Proof. Suppose that v satisfies (DGPW). Then it is clear that v satisfies (PW), provided we can show that v satisfying the Rayleigh expansion radiation condition implies that v satisfies the UPRC (2.5). But this is shown in 8]. Conversely, suppose that v satisfies (PW). Then

$$v \quad x_1$$

Proof. It is almost immediate from the observations immediately above the theorem that if v satisfies (PWSC) then u, defined by (5.5), satisfies the above boundary value problem. The only difficulty is to show the radiation condition. To see this we note that v^{sc} satisfies the radiation condition (2.5), from which it follows (see 9] and cf. Remark 2.2) that v^{sc} satisfies (2.5) with $_0$ replaced with $_c$, for all $c >$, in particular with $c = -b$. Since $u = v^{sc}$ in U_c it is immediate that v satisfies (2.5) with $_0$ replaced by $_{-b}$, which is equivalent (see Remark 2.2) to (2.4) with x_2 replaced by $x_2 + b$.

We next observe that it follows from Theorem 4.1 that the boundary value problem for u has exactly one solution (u satisfies exactly a boundary value problem of the form of Section 2 after vertical translation of the axes by a distance $|b|$). The theorem is thus proved if we can show that this solution satisfies that $u|_{S_h} \in V_h^\infty$, for every $h >$, and the bound $|u|_{S_h}|_{V_h^\infty} \leq C_p$.

To see this we make the follo

where $\mathbf{G} \in \mathbf{V}^{\square}_{-\square}$ *is defined by*

$$\mathbf{G}\,\mathbf{w} = \int_{\Gamma_0} \mathbf{y}_-\mathbf{w} \left(\frac{\partial \mathbf{v}^{in}}{\partial \mathbf{x}_2} + \mathbf{T}\mathbf{y}_-\mathbf{v}^{in}\right) \mathbf{ds} \quad , \quad \mathbf{w} \in \mathbf{V}_{-\square}. \tag{5.8}$$

The restriction to the range $\square < - \,/$ *arises since* $\mathbf{v}^{in} \in \mathbf{V}_{\square}$ *for* $\square < - \,/$ *but not for* $\square = \,/$. *Having solved this variational problem to determine* $\mathbf{v}|_{\mathbf{S_0}}$, $\mathbf{v}$ *is determined throughout* $\mathbf{D}$ *through* (2.5) *satisfied by* $\mathbf{v}^{sc}$. *Of course this variational formulation is well-posed, by Theorem 4.1.*

Remark 5.5. *The above results show that the problem of plane wave incidence is well posed in the 2D case. In the 3D case it seems to us likely that a solution to the problem of plane wave incidence does not exist for every choice of domain* $\mathbf{D}$ *satisfying* (2.1) *and* (2.2). *Certainly, the methods of argument above do not extend to the 3D case, for, in the 3D case,* $\mathbf{g_P}$ *in Theorem 5.3 is in* $\mathbf{L}^2_{\square}\,\mathbf{D}$ *only for* $\square < -$, *and* $\mathbf{G}$ *given by* (5.8) *is in* $\mathbf{V}^{\square}_{-\square}$ *only for* $\square < -$, *so that Theorem 4.1 does not apply. Further, even the formulation of the 3D plane wave problem appears problematic in 3D. Precisely, just as the radiation condition* (2.4) *does not extend to a bounded linear functional on* $\mathbf{H}^{1/2}_{\square}\;_0$ *for* $\square < -$, *it does not extend to a bounded linear functional on* $\mathbf{L}^{\infty}\;_0$ *(which would require that the integral in (2.5) be absolutely convergent for every* $\mathbf{u}_0 \in \mathbf{L}^{\infty}\;_0$, *which is true in 2D but not in 3D, as a consequence of the asymptotics* (2.8)*). Thus it is* *where the constants* A *and* B *are chosen to ensure that* $v^{in} \in C^1(\mathbb{R}^2)$ *(again this is possible provided* ϵ *is chosen sufficiently small). Then* $v^{in} \in H^2_{loc}(\mathbb{R}^2)$ *with* $(\Delta + k^2)v^{in} = g_c$*, where* $g_c(x) = Ak^2$, $\sqrt{x_1^2 + (x_2 - H)^2} < \epsilon$, $g_c(x) = 0$*, otherwise. We observe that* g_s *is compactly supported so that* $g_s \in L^2_\rho(D)$ *for every* $\rho \in \mathbb{R}$*. Further, it is an easy calculation to see that* $g_c \in L^2_\rho(D)$ *for* $\rho < -1/2$*,* explicitly $V_\varrho^{(R)}$ denotes the completion of $\{u|_{S_0^{(R)}} : u \in C_0^\infty(D^{(R)})\}$ in the norm

$$\|u\|_{V_\varrho^{(R)}} = \left(\int_{S_0^{(R)}} \left(\left|(1 + x_2^2)^{\varrho/2} u\right|^2 + \left|\nabla\left((1 + x_2^2)^{\varrho/2} u\right)\right|^2\right) dx\right)^{1/2}. \tag{5.12}$$

We remark, as is easily seen from Lemma 2.1, that the norms $\|\cdot\|_{V_\varrho^{(R)}}$, $\varrho \in \mathbb{R}$, are equivalent since $S_0^{(R)}$ is bounded, so that, as linear spaces, for $\varrho \in \mathbb{R}$, $V_\varrho^{(R)} = V^{(R)} = V_0^{(R)}$. The approximating variational problem is the following: find $u^{(R)} \in V^{(R)}$ such that

$$B^{(R)}(u^{(R)}, v) = -(g, v), \quad \forall v \in V^{(R)}. \tag{5.13}$$

Here $B^{(R)}$ is the continuous sesquilinear form on $V^{(R)} \times V^{(R)}$ defined by (3.3) with D replaced by $D^{(R)}$, i.e. defined by

$$B^{(R)}(u, v) = \int_{S_0^{(R)}} \left(\nabla u \cdot \nabla \bar{v} - k^2 u \bar{v}\right) dx + \int_{\Gamma_0^{(R)}} \gamma_- \bar{v}\, T\gamma_- u\, ds, \tag{5.14}$$

where $\Gamma_0^{(R)} = \overline{S_0^{(R)}} \cap \Gamma_0$ (see Remark 3.6 for the interpretation of γ_- in this case).

Making the observation that we can view $V_\varrho^{(R)}$ as a closed subspace of V_ϱ (the elements of $V_\varrho^{(R)}$ become elements of V_ϱ if we extend them by zero from $S_0^{(R)}$ to S_0), the analysis of the error in approximating u by $u^{(R)}$ follows the usual pattern for analysing the Galerkin method for variational problems via a generalized Céa's lemma. Precisely, if $u \in V^{(R)} \subset V_\varrho$, then, for $v \in V^{(R)}$, applying (5.11),

$$B^{(R)}(u, v) = B(u, v) = B(u - u, v) - (g, v).$$

Subtracting this equation from (5.13) we see that

$$B^{(R)}(u - u^{(R)}, v) = B(u - u, v), \quad \forall v \in V^{(R)}. \tag{5.15}$$

Now recall from Section 4 that $\mathcal{B}_\varrho : V_\varrho \to V_{-\varrho}^*$ is our notation for the bounded linear operator constants dependent only on □ and $|\mathbf{b}|$,

$$\|\mathbf{u}-\mathbf{u}\|_{\mathbf{V}_{\varrho_1}} = \| \quad - \boldsymbol{\chi}_{\mathbf{R}} \ \mathbf{u}\|_{\mathbf{V}_{\varrho_1}}$$
$$\leq \mathbf{c}_2 \left(\int_{\tilde{\mathbf{S}}_{\mathbf{0}}^R} \quad + \mathbf{x}^2 \ \square_{\mathbf{1}} \right.$$

THEOREM 6.1. *For* $ka \geq$ *and* $|\square| <$, *the commutator* C_a *defined in* (6.3) *has norm* $\leq c\,\square\,\sqrt{k/a}$ *on* $L^2\ \mathbb{R}^m$.

It is enough to consider $\square \in\ ,$ since the case of negative $\square$ then follows by duality (with respect to the scalar product on $L^2\ \mathbb{R}^m$). We split the symbol t_a as

$$t_a = t^{(0)} + t^{(1)} = \ \chi\ |\xi|\ \ t_a\ \xi\ +$$

with $\mathbf{b}_{□} = \mathbf{F} \quad + \mathbf{x}^2 \; ^{-□/2}$. Here the integral in (6.14) is well defined since $\mathbf{Fu}$ is rapidly decreasing and $\mathbf{b}_{□} \in \mathbf{L}^1 \; \mathbb{R}^{\mathbf{m}}$ for $□ >$ (see the next lemma), and we have used the relation $\mathbf{F} \quad + \mathbf{x}^2 \; ^{-□/2}\mathbf{v} = \mathbf{b}_{□} * \mathbf{Fv}$ for a function $\mathbf{v}$ of rapid decay, with $*$ denoting convolution.

LEMMA 6.4. *For any* $□ >$ *, the functions* $\mathbf{b}_{□}$ *and* $|\boldsymbol{\xi}| \nabla_{\boldsymbol{\xi}} \mathbf{b}_{□}$ *are rapidly decreasing as* $|\boldsymbol{\xi}| \to \infty$ *and belong to* $\mathbf{L}^1 \; \mathbb{R}^{\mathbf{m}}$ *.* For the proof of this, we refer to 29, Chap. 8.1]; see also 34, Chap. 5.3].

Proof of Theorem 6.2 (i). From (6.14) and Lemma 6.3,

$$\|\mathbf{NFu}\|_{\mathbf{L}^2(\mathrm{R}^m)} \leq \left\| \int_{\mathrm{R}^m} |\mathbf{b}_{□} \; \boldsymbol{\xi} - \boldsymbol{\eta} \; | \, |\boldsymbol{\xi} - \boldsymbol{\eta}| \quad _{\mathrm{R}\chi} \mathrm{p} \right.$$

where $\boldsymbol{\phi}$ is a smooth function with somewhat larger support and $\boldsymbol{\phi\psi} = \boldsymbol{\psi}$.

Let first $\mathbf{m} =$. Then (6.20) follows for □ $\in$

By taking Fourier transform, the uniform boundedness of (6.27) is equivalent to the estimates

$$\|\mathbf{m}\ \mathbf{x_n},\boldsymbol{\xi}\ \mathbf{v}\|_{\mathbf{H}^{\varrho}(\mathbb{R}^m)} \leq \mathbf{c}\ \mathbf{h}, \ \|\mathbf{v}\|_{\mathbf{H}^{\varrho}(\mathbb{R}^m)}, \quad \mathbf{v} \in \mathbf{C}_0^{\infty}\ \mathbb{R}^{\mathbf{m}}\ , \quad \mathbf{x_n} \in\ \ ,\mathbf{h}\ , \tag{6.29}$$

where $\mathbf{m}\ \mathbf{x_n},\boldsymbol{\xi}\ =\ \ \mathrm{p}\ -\mathbf{x_n}\mathbf{t}\ \boldsymbol{\xi}$. Consider a decomposition $\mathbf{t} = \mathbf{t}^{(0)} + \mathbf{t}^{(1)}$ as in (6.5), with $\mathbf{a} =\ $, $\mathbf{t}^{(0)} = \boldsymbol{\chi}\mathbf{t}$, $\mathbf{t}^{(1)} =\ \ -\boldsymbol{\chi}\ \mathbf{t}$ and a cut-off function $\boldsymbol{\chi}$ vanishing near $|\boldsymbol{\xi}| = \mathbf{k}$, so that $\mathbf{t}^{(0)}$ is a smooth symbol. We

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