School of Mathematics, Meteorology
and Physics

Department of Mathematics

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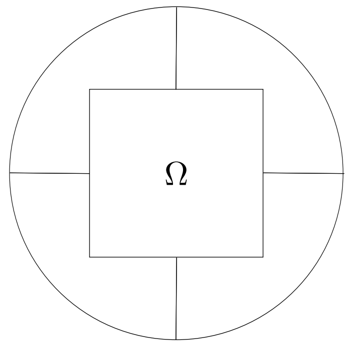
An exponentially convergent nonpolynomial finite element method for time-harmonic scattering from polygons

by

A.H. Barnett and T. Betcke


Ω

argument we choose $\eta = k$. The space of all linear combinations of the fundamental solutions in Ω_e^+ is denoted by V_e.

REMARK. One could alternatively approximate the scattered field u_s by a multipole expansion of the form

$$u_s(r, \theta) \approx \sum_{m=-N_e}^{N_e} \alpha_m H_m^{(1)}(kr) e^{im\theta} .$$

This was proposed by Stojek in [22]. The disadvantage of this expansion is that when Γ_e is anything other than a circle about the origin, severe numerical stability problems arise at large wavenumber due to the huge dynamic range of Hankel functions at large m. Fundamental solutions do not suffer from this problem, hence allow more flexibility (e.g. see Section 6.2).

Combining the above basis sets, the trial space V of the FEM is the space of functions v such that $v_i := v|_{E_i} \in V_i$, $\forall i$, and $v_e := v|_{\Omega_e^+} \in V_e$. It is useful to express the number of basis functions in each subdomain as a multiplier of a common factor N. Let $N_i = n_i N$, We now derive a matrix representation of $J(c)$. Consider first the internal boundary Γ_{ij} between two elements E_i and E_j, within which the basis functions are $g_1^{(i)},\dots,g_{N_i}^{(i)}$ and $g_1^{(j)},\dots,g_{N_j}^{(j)}$. We assume these functions and their derivatives are also defined on Γ_{ij}. Furthermore, denote by ξ , $\tau = 1,\dots,m_{ij}$ quadrature points on Γ_{ij} with corresponding weights ω > 0 appropriate for integration with respect to arc length. Then from Γ_{ij} the contribution to $J(c)$ is

$$\int_{\Gamma_{ij}} k^2 \sum_{p=1}^{N_i} c_p^{(i)} g_p^{(i)}(s) - \sum_{p=1}^{N_j} c_p^{(j)} g_p^{(j)}(s) \;{}^2 + \sum_{p=1}^{N_i} c_p^{(i)} \partial\ g_p^{(i)}(s) - \sum_{p=1}^{N_j} c_p^{(j)} \partial\ g_p^{(j)}(s) \;{}^2\ ds$$

$$\approx \quad \begin{bmatrix} W_{ij} & \\ & W_{ij} \end{bmatrix} \begin{bmatrix} kA_i & -kA_j \\ \partial\ A_i & -\partial\ A_j \end{bmatrix} \begin{bmatrix} c^{(i)} \\ c^{(j)} \end{bmatrix} {}_2^2,$$

where for each $i = 1,\dots,q$, A_i is the matrix with elements $(A_i)\ _p = g_p^{(i)}(\xi\ )$, and $(\partial\ A_i)\ _p = \partial\ g_p^{(i)}(\xi\ )$ is the matrix of normal derivatives. Quadrature weights now reside in the diagonal matrix W_{ij} with elements $(W_{ij})\ \ = \omega^{1\ 2}$, τ

Let D be a simply connected domain with analytic boundary ∂D. Let u_s be the unique solution [9] to the Helmholtz equation (1.1) in $\mathbb{R}^2 \backslash D$ and the Sommerfeld condition (1.3) with boundary value data $u_s = f$ on ∂D. The idea of the MFS is to choose a closed curve Γ_F D some distance inside D, and approximate u_s by a single-layer potential of the form

$$u_s(x) \approx \int_{\Gamma_F} \frac{i}{4} H_0^{(1)}(k|x - y|)g(y)ds_y.$$

We now claim that the eigenvalues $\hat{s}_\ell(m)$ and their derivatives with respect to the outer disc radius r decay exponentially with a rate that depends on the radius ratio of R to r.

LEMMA 4.[illegible]. For $\eta = 0$ and $\epsilon > 0$ arbitrarily small there exist constants $c_s > 0$ and $C_s > 0$ such that for $m \in \mathbb{Z}$ both the following hold,

$$c_s \left(\frac{r}{R}\right)^{-|\ell|} \le |\hat{s}_\ell(m)| \le C_s \left(\frac{r}{R}\right)^{-|\ell|}, \tag{4.3}$$

$$\frac{\partial}{\partial r}\hat{s}_\ell(m) \le C_s \left[\left(\frac{r}{R}\right) - \epsilon\right]^{-|\ell|}, \tag{4.4}$$

where c_s and C_s depend on k, R and r, and C_s additionally depends on ϵ.

Proof. Consider the three terms in (4.2). Large-order asymptotics for Bessel functions [1, 9.3.1] yield

$$J_\ell(kR) \sim \frac{1}{\sqrt{2\pi m}}\left(\frac{ekR}{2m}\right)^{\ell}, \quad Y_\ell(kr) \sim -\sqrt{\frac{2}{\pi m}}\left(\frac{ekr}{2m}\right)^{-\ell}$$

for fixed z and $m \to \infty$. Since $H_\ell(z) = J_\ell(z) + iY_\ell(z)$ it follows that $H_\ell(z) \sim iY_\ell(z)$ and therefore

$$\begin{aligned}
H^{(1)}_\ell(kr)J_\ell(kR) &\sim -\frac{i}{\pi m}\left(\frac{R}{r}\right)^{\ell},\\
H^{(1)}_\ell(kr)J_{\ell-1}(kR) &\sim -\frac{2i}{ekR\pi}\left(\frac{R}{r}\right)^{\ell},\\
H^{(1)}_\ell(kr)J_{\ell+1}(kR) &\sim -\frac{ekRi}{2\pi}\frac{1}{m^2}\left(\frac{R}{r}\right)^{\ell}.
\end{aligned}$$

Inserting these into (4.2) we get

$$\hat{s}_\ell(m) \sim \frac{1}{2eR}\left(\frac{R}{r}\right)^{\ell} + \frac{i\eta}{2m}\left(\frac{R}{r}\right)^{\ell} - \frac{1}{2eR}\left(\frac{R}{r}\right)^{\ell}.$$

Together with the reflection laws $J_{-\ell}(z) = (-1)^\ell J_\ell(z)$ and $H^{(1)}_{-\ell}(z) = (-1)^\ell H^{(1)}_\ell(z)$ the upper

We will also need the Fourier series for this choice of g, which we note converges only distributionally. Denote by $\hat{c}$ the discrete Fourier transform of the coe□cient vector $c = c_1, \ldots, c_N$ defined by

$$\hat{c}_s = \frac{1}{N}\sum_{j=1}^{N} c_j e^{-is\ \ j}, \quad -\frac{N}{2} < s \quad \frac{N}{2}.$$

Then applying our ansatz to the definition of the Fourier coe□cents,

$$\hat{g}(m) = \frac{1}{2\pi}\sum_{j=1}^{N} c_j e^{i\ \ j} = \frac{N}{2\pi}\hat{c}_{(\ \ \mathrm{mod}\ N)}, \tag{4.7}$$

where m mod N denotes the unique integer lying in the range $-N/$

as N_i . Furthermore, there exist functions $\tilde{v}_\bullet$ V_i such that $u - \tilde{v}$ $_{L^\infty(E_i)} = O(\tau^{-N_i})$ and also

$$u - \tilde{v}\ _{L^\infty(\Gamma_{ij})} = O(\tau^{-N_i})$$

for every $1 < \tau < \rho_i$ as N_i .

The rates ρ_i may be computed; they are the conformal distance of the nearest singularity in u to (a conformal map of) the domain E_i [5].

To estimate the convergence on Γ_e of the fundamental solutions approximation to the scattered field u_s we consider (as in Section 4) only the case of concentric circles.[2] The source points for the fundamental solutions are given by $y_j = Re^{i\ j}$, $j = 1, \ldots, N_e$ with $\varphi_j = \frac{2\ j}{N_e}$, and the exterior circle Γ_e has radius r. We impose $\max_i p_i < R < r$. The proof of the following theorem is given below in Section 5.1.

THEORE . Let > 0 and > 0 be arbitrarily small. Define $\rho := \min_i \frac{r}{|p_i}$

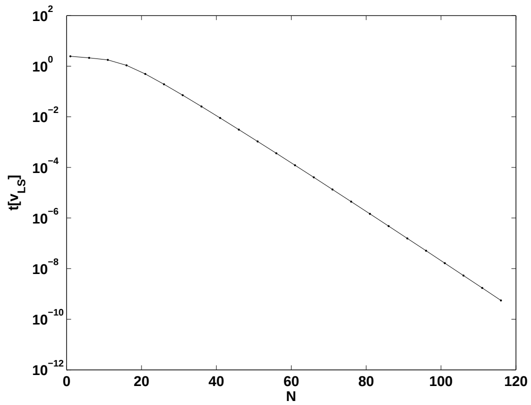
Using (5.4) we can now estimate E_u^2 as

$$E_u^2 \quad C \sum_{\ell \notin [-\frac{N_e}{2}+1, \frac{N_e}{2}]} \rho^{-2|\ell|} \quad C\rho^{-N_e}. \tag{5.7}$$

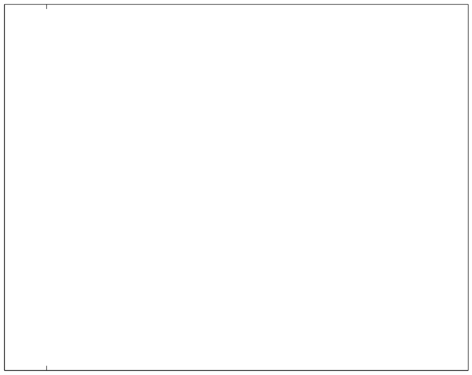
In order to estimate E_s^2 we rewrite it as

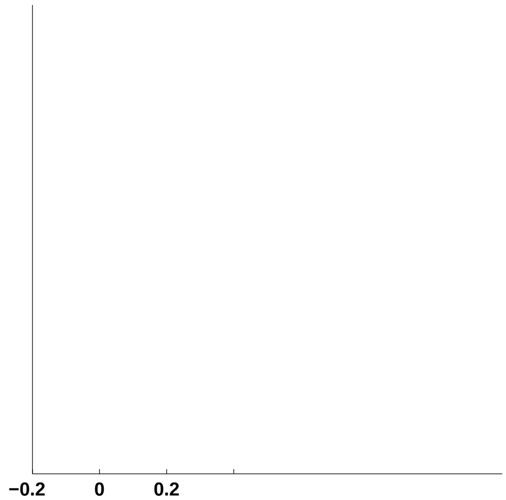
$$E_s^2 = \sum_{n=-\frac{N_e}{2}+1}^{\frac{N_e}{2}} \hat{g}(n)^2 \sum_{\ell \in \mathbb{Z}\setminus\{0\}} \hat{s}(bN_e + n)^2.$$

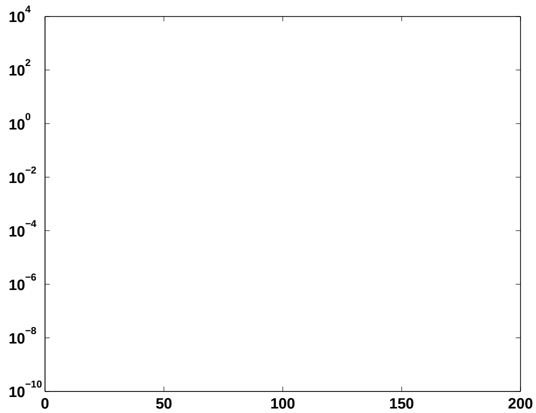
Bounding the inner sum and substituting $\hat{g}(n) = \frac{\hat{u}_s(n)}{\hat{s}(n)}$



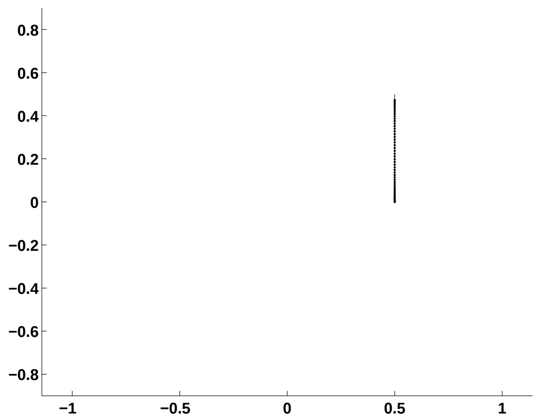
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
$t[v_{LS}]$
0
20
40
60
80
100
120
N




−0.2
0
0.2



10^{4}
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
0
50
100
150
200



0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−0.5
0
0.5
1

LE[illegible]. Let $\tilde{c}_{LS}$ be defined by the above perturbed problem. Then it holds that

$$t_d[v_{LS}] \le t_d[\tilde{v}_{LS}] \le t_d[v_{LS}] + C(2 + \|c_{LS}\|_2 + \|\tilde{c}_{LS}\|_2)\epsilon_{mach} \tag{7.2}$$

Proof. Using the property that c_{LS} minimizes the unperturbed least-squares problem, and the triangle inequality, results in

$$\|WAc_{LS} - Wb\|_2 \le \|WA\tilde{c}_{LS} - Wb\|_2 \le \|(WA + E)\tilde{c}_{LS} - (Wb + f)\|_2 + C(1 + \|\tilde{c}_{LS}\|_2)\epsilon_{mach}.$$

Exchanging perturbed and unperturbed quantities gives similarly

$$\|(WA + E)\tilde{c}_{LS} - (Wb + f)\|_2 \le \|WAc_{LS} - Wb\|_2 + C(1 + \|c_{LS}\|_2)\epsilon_{mach}.$$

Combining the two estimates shows (7.2). $\square$

Thus, if the coefficient norms at the approximate and exact minima are small, the numerical least-squares solution must converge at the same exponential rate (Theorem 5.4) as the exact least-squares solution.

In [4] the blow-up of the coefficient vector of fundamental solutions approximations was investigated for interior Helmholtz problems. Now we prove an analogous theorem that will help us choose a numerically useful fundamental solutions curve.

THEORE[illegible]. Consider a sequence of fundamental solutions approximations, each of the form $v(x) = \sum_{j=1}^{N_e} c_j^{(e)} \left(\frac{i}{4} \frac{[illegible]}{(y_j)} H_0^{(1)}(k|x - y_j|) - \frac{[illegible]}{4} H_0^{(1)}(k|x - y_j|)\right)$, with growing numbers N_e (N_e even) of charge points, that attains the error bound from Theorem 5.2 as $N_e \to \infty$. Let each coefficient vector be written $c^{(e)} := \left[c_1^{(e)}, \ldots, c_{N_e}^{(e)}\right]^T$. If $R > \max_i |p_i|$, then the sequence of norms $\|c^{(e)}\|_2$ is bounded independently of N_e.

Proof. In the following $C > 0$ denotes an unspecified constant that depends on k, R, r and [illegible] but not on N_e and may change throughout the proof. By assumption $\max_i |p_i| < R < r$.

From (5.2) and (5.5) it follows that

$$\left|k\sqrt{2\pi r}\,\hat{u}_s(m) - \hat{s}(m)\hat{g}(m)\right| \le C\tau^{-N_e}, \quad m \in \mathbb{N}$$

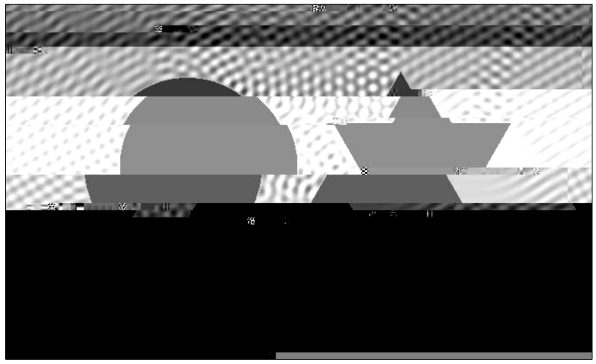
where $\tau = \min\left\{\left(\frac{r}{R}\right), \rho^{1/2}\right\}$ (note that by Remark 5.3, [illegible] = 0 in Theorem 5.2 is possible since we only use the estimate for the function and not for the normal derivative). It follows that

$$|\hat{g}(m)| \le |\hat{s}(m)|^{-1}\left(C\tau^{-N_e}(2\pi r)^{-1/2} + \hat{u}_s(m)\right), \quad m \in \mathbb{N}$$

Now restrict m to the interval $[-N_e/2 + 1, \ldots, N_e/2]$. If $R > \sqrt{r \max_i |p_i|}$ then for sufficiently [illegible]

From (4.7) it follows that $\hat{g}(m) =$

Full Field (Real Part)



```
% Exponentially accurate sound-soft time-harmonic scattering from the square
k = 50;                 % Wavenumber
r = 1.0;                % Radius of outer artificial circle
M = 100;                % Number of quadrature points on each segment
N = 90;                 % Number of basis funcs in each corner subdomain
a = 0.5;                % Half-size of the square
R = sqrt(0.5);          % Radius of the fundamental solutions curve

% Define segments...
s = segment.polyseglist(M, [1i*r 1i*a a+1i*a a r]);          % straight
s = [s(1:3) segment(3*M, [0 r 0 pi/2])];                     % add arc
s = [s rotate(s, pi/2) rotate(s, pi) rotate(s, 3*pi/2)];    % add 3 copies
sart = s([1 4 5 8 9 12 13 16]);       % list of all artificial boundaries
sext = s([4 8 12 16]);                % segments forming outer circle

% Define domains...
for j=1:4, d(j) = domain(s(1+mod(4*(j-1)+[0 1 2 12 3],16)),[1 1 1 -1 1]); end
ext = domain([], [], sext(end:-1:1), -1);
sart.setmatch([k -k], [1 -1]);        % matching conditions between elements

% Basis functions...
nuopts = struct('type','s', 'cornermultipliers',[0 0 1 0 0], 'rescale_rad',1);
for j=1:4, d(j).addcornerbases(N,nuopts); end                % frac-order FB
Z = @(t) R*exp(2i*pi*t); Zp = @(t) 2i*pi*R*exp(2i*pi*t);    % fund soln curve
ext.addmfsbasis({Z, Zp}, N, struct('eta',k, 'fast',2, 'nmultiplier',2.1));

p = scattering(ext, d);               % now set up problem, solve, and plot...
p.setoverallwavenumber(k);
p.setincidentwave(-pi/6);
p.solvecoeffs;
fprintf('least-square err = %g, coeff norm = %g\n', p.bcresidualnorm, norm(p.co))
p.showfullfield(struct('bb',[-1.5 1.5 -1.5 1.5], 'dx',0.02));
```

FIG. A. **MATLAB code for sound-soft scattering from a square using** MPSpack **toolbox.**

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