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Extreme Functionals and Stone-Weierstrass Theory of Inner Ideals in JB*-Triples

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Abstract

Let I and J be norm closed inner ideals of a JB*-triple. The main theorem of the thesis states that I and J are equal precisely when $\partial_e(I_1^*) = \partial_e(J_1^*)$. Moreover, we prove that $I \subset J$ exactly when $\partial_e(I_1^*) \subset \partial_e(J_1^*)$. Thus, JB*-triple inner ideals are determined by extreme dual ball points.

The tool used to reach this conclusion is what we term the Inner Stone-Weierstrass Theorem for JB*-triples; we show that for norm closed inner ideals I and J of a JB*-triple, where $I \subset J$, we may conclude that $I = J$ if $\partial_e(I_1^*) = \partial_e(J_1^*)$. Our excuse for this terminology is that the equality of the extreme dual ball points implies a Stone-Weierstrass separation condition, that is, that I separates $\partial_e(J_1^*) \cup \{0\}$.

To create this tool, we first exploit structure space techniques to make a

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Contents

Introduction

Historically, JB*-triples originate in the study of an algebraic characterisation of bounded symmetric domains in complex Banach spaces [Kaup]. Examples of JB*-triples include C*-algebras and JB*-algebras.

Given a norm closed inner ideal I of a JB*-triple A, it is well-known that each functional in $\partial_e(I_1^*)$ has a unique extension to a functional in the dual ball of A, and that in fact this extension is an extreme point. In addition, there is a bijective correspondence between $\partial_e(A_1^*)$ and the minimal tripotents of A^{**} [FrRu4]. Therefore, by identifying each functional $\rho \in \partial_e(I_1^*)$ with its extension, we may write

$$\partial_e(A_1^*) = \bigcup \partial_e(I_1^*);$$

as the union ranges over all norm closed inner ideals I of A.

The purpose of this thesis is to investigate to what extent the inner ideal structure of a JB*-triple is determined by these extreme functionals. We show that two norm closed inner ideals I and J of a JB*-triple A are equal precisely when $\partial_e(I_1^*) = \partial_e(J_1^*)$, and furthermore, that $I \subset J$ if and only if $\partial_e(I_1^*) \subset \partial_e(J_1^*)$.

Predominantly, however, the thesis is concerned with deriving the tool used to reach thes(72448(determisatis,Tf5.84x5ued)-(7b85.21x5ued)-(7p4x5ued)reac)2o[h5bnn9woIo wha

Essentially, the route we take to prove the Inner Stone-Weierstrass Theorem is to progressively establish the analogous result for various triple structures; for C*-algebras and universally reversible JC*-algebras in Chapter Four, and then for JC*-triples and JB*-triples in Chapter Five, with each stage intrinsically reliant upon the preceding one. A pivotal step, and one of possible independent interest, is the formation of a particular composition series that allows the extension of the universally reversible JC*-algebra version to its JC*-triple counterpart. Thus, we prove:

If A be a JC-triple such that all Cartan factor representations of A have rank greater than two, then A has a composition series of norm closed ideals* $(J_{¸})_{0 \cdot ¸ \cdot ®}$ *such that* $J_{¸+1}=J_{¸}$*, is isomorphic to an inner ideal in a universally reversible JC*-algebra.*

It should be understood that the opening two chapters comprise of well-known material, presented to support subsequent chapters. The emphasis here is on clarity and brevity. In Chapter One we supply the principal background to Jordan algebras, whereas Chapter Two details JB*-triples. We include known results relating to the focus of the thesis, that is JB*-triple inner ideals, alongside some original, wholly technical lemmas regarding these structures. Given our choice of terminology, we state brief details of the Stone-Weierstrass Theorem and its many generalisations.

The third chapter marks the beginning of the main body of original work. In essence, our aim is to undertake the groundwork required for what follows, focussing upon certain aspects of inner ideals in JW*-triples. In particular, by making extensive use of Horn's decomposition theory, [Ho2], we demonstrate a strong correspondence between the underlying generic type of homogenous JW*-triples and that of their weak* closed inner ideals. In this endeavour we also exploit the powerful notion of the centroid of a JB*-triple [DinTi][EdRü9].

Moving on, we investigate the inner ideal structure of universally reversible JW*-algebras. Our starting point is the work of [EdRüVa2]. For continuous JW*-algebras, or those isomorphic to von Neumann algebras, M say, the authors proved that each inner ideal is of the form $eM\phi(e)$, where ϕ is the canonical involution of the enveloping von Neumann algebra. It was our intention to form a definitive resolution using this template, however our extension is valid only for those algebras without symplectic part. Although this exception to some extent impedes our work, the restricted resolution is sufficient for the needs of the thesis.

In Chapter Four we prove the Inner Stone-Weierstrass Theorem for universally reversible JC*-algebras:

If A is a universally reversible JC-algebra with norm closed inner ideals I and J, with I contained in J and such that $\partial_e(I_1^*) = \partial_e(J_1^*)$, then $I = J$.*

We begin with a series of technicalities regarding the atomic part of a JB*-triple, that are used extensively throughout. We show that if A is a JB*-triple with a norm closed inner ideal I, then if $\partial_e(I_1^*) = \partial_e(A_1^*)$ we can conclude that I^{**} and A^{**} have equal atomic part, denoted by A_{at}^{**}. Furthermore, we prove that $I_{at}^{**} = A_{at}^{**}$ precisely when I separates $\partial_e(A_1^*) \cup \{0\}$. The latter is a Stone-Weierstrass separation condition. Thus our choice of nomenclature.

Chapter 1

Preliminaries of Jordan Algebras

1.1 Introduction

In this chapter we lay out the principal preliminaries on Jordan algebras

Particular attention is paid to the concepts of type I decomposition and factor representations. These are recurrent themes within the thesis and therefore merit a clear treatment. In a similar vein we highlight the notions of universal reversibility and the universal enveloping C*-algebra.

We conclude by introducing JB*-algebras, a key part of what follows. Here a brief definition will suffice since, for the most part, necessary remarks can be derived from our exposition of JB-algebras.

We use standard notation, thus given a Banach space X, X^* denotes the dual space. We will habitually regard X as being contained in the second dual via the canonical embedding. In the same way, X^* is contained in X^{***}. The transpose of the embedding $X \mapsto X^{**}$ is the weak* continuous map $P : X^{***} \to X^*$, $(\rho \to \rho|_{X^*})$. If X is the dual of some Banach space Y then this map is $P : Y^{**} \to Y$, the weak* continuous projection.

We shall make frequent and tacit use of the following well-known result. Let X and E be Banach spaces, where E is the dual of some Banach space Y. Let $\pi : X \to E$ be a bounded linear map such that $\pi(X)$ is weak* dense in E. Then there is a unique weak* continuous extension, $\tilde{\pi} : X^{**} \to E$, and, moreover, $\tilde{\pi}(X^{**}) = E$.

Finally, as is usual, we will use $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ to denote the natural, real and complex numbers, and $\mathbb{H}$ and $\mathbb{O}$ to denote the quaternions and the octonions respectively.

1.2 Jordan Algebras

We now describe a specific class of algebras, namely Jordan algebras.

1.2.1 Let A be an algebra over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, with product

$$\circ : A \times A \to A$$

denoted by $a \circ b$, for $a, b \in A$. A is said to be a *Jordan algebra* if it satisfies the following two properties:

(i) $a \circ b = b \circ a$ for all $a, b \in A$.

(ii) $(a \circ b) \circ a^2 = a \circ (b \circ a^2)$ for all $a, b \in A$.

1.2.2 Let A be an associative algebra. A new product, $\circ$, called the *special Jordan product*, on A is defined by

$$a \circ b = \frac{1}{2}(ab + ba),$$

where ab denotes the usual product. The product $\circ$ is bilinear and commutative. Let A^J denote the algebra A endowed with this product $\circ$. Under the special Jordan product the conditions (1.2.1(i)) and (1.2.1(ii)) given earlier hold, so that A^J is a Jordan algebra. A *special Jordan algebra* is a Jordan algebra that is isomorphic to a subalgebra of A^J, for some associative algebra A.

1.2.3 There exist Jordan algebras which are not special, i.e. that are not Jordan subalgebras of associative algebras with the special Jordan product. These are called *exceptional Jordan algebras*. A classical example is the 27 dimensional octonionic real Jordan algebra $M_3(\mathbb{O})_{sa}$, which we will denote by N_3^8.

1.2.4 We now define two fundamental operators on Jordan algebras. Let A be a Jordan algebra. For all elements a of A define the

1.3.2 A *JB-algebra* is a real Jordan algebra A with a norm $k{:}k$ such that

(J1) k

1.4.1 A Jordan algebra A is said to be *associative* or *abelian* if $A = Z(A)$. Let X be a locally compact Hausdorff space. Then the self adjoint part of $C_0(X)$ is an associative JC-algebra under pointwise multiplication and supremum norm. Conversely we have the following.

Theorem 1.4.2 ([AlShSt, 2.3])

Let A be an associative JB-algebra. Then there exists a locally compact Hausdorff space X such that A is isometrically isomorphic to $(C_0(X))_{sa}$. Moreover A is unital if and only if X is compact.

1.4.3 Let A be a JB-algebra with $a_1, \ldots, a_n \in A$ and let $C(a_1 \ldots a_n)$ denote the JB-subalgebra of A generated by $a_1, \ldots, a_n$. Then if $a_1, \ldots, a_n$ operator commute $C(a_1 \ldots a_n)$ is abelian and hence is isometric to the self adjoint part of some abelian C*-algebra. In particular, via C*-algebra theory, for any $a \in A$ we have the following isometric isomorphism

$$\begin{aligned} C_0(\sigma(a)) &\cong C(a) \\ f &\mapsto f(a), \end{aligned}$$

where $\sigma(a) = \{\lambda \in \mathbb{R} : a - \lambda 1 \text{ is not invertible in } C(1, a)\}$. Here, if A is non-unital 1 is the identity element of the unitisation of A. Note that in Jordan algebra terms a is said to be *invertible* with inverse b if $a \circ b = 1$ and $a^2 \circ b = a \circ b$.

1.4.4 Let A be a JB-algebra and let $a \in A$. Then a is said to be *positive*

1.5 States

1.5.1 Let A be a JB-algebra. A functional $\rho \in A^*$ is *positive* if $\rho(a) \geq 0$ for all $a \in A_+$. In which case we write $\rho \geq 0$.
The set of *quasi states* of A,

$$Q(A) = \{\rho \in A^* : \rho \geq 0, \|\rho\| \leq 1\},$$

is weak* compact and convex. The set of states of A is the convex set

$$S(A) = \{\rho \in A^* : \rho \geq 0, \|\rho\| = 1\}.$$

The non-zero extreme points of $Q(A)$ are states of A called the *pure states* of A, the set of which is denoted by $P(A)$. Furthermore, if A has an identity element, denoted by 1, then

$$S(A) = \{\rho \in A^* : \rho(1) = \|\rho\| = 1\}$$

and is weak* compact as well as being convex. Consequently, in that case we have $P(A) = \partial_e(S(A))$, that is, the set of extreme points of $S(A)$.

1.6 JW-Algebras and JBW-Algebras

It is natural to consider a subclass of JB-algebras which are in some sense the Jordan analogue of W*-algebras. We begin with the concrete version, JW-algebras.

1.6.1 Let H be a complex Hilbert space and consider $B(H)$ as a von Neumann algebra with the weak topology. A real Jordan subalgebra M of $B(H)_{sa}$ is said to be a *JW-algebra* if it is a weakly closed.

1.6.2 Before we can formally define JBW-algebras we need a few preliminary definitions. Let M be a JB-algebra. Then M is said to be *monotone complete* if each bounded increasing net (x_α) in M has least upper bound x in M. A bounded linear functional ρ of M is said to be *normal* if for every such net (x_α) we have $\rho(x_\alpha) \longrightarrow \rho(x)$. A set of functionals Γ is said to be *separating* if for any non-zero x in M there exists a functional ρ in Γ such that $\rho(x) \neq 0$.

1.6.3 A JB-algebra M is said to be a *JBW-algebra* if it is monotone complete with a separating set of positive normal bounded linear functionals. In line with Sakai's definition of a W*-algebra, there is an alternative and more elegant definition.

Theorem 1.6.4 ([Sh1, Theorem 2.3])

Let M be a JB-algebra. Then M is a JBW-algebra if and only if M is a Banach dual space.

In this case the predual is unique and consists of the normal linear functionals on M. It is denoted by M_*.

1.6.5 Every JW-algebra is a JBW-algebra. All JBW-algebras are unital [HaSt, 4.1.7].

1.6.6 The *normal states* of a JBW-algebra M are the weak* continuous linear functionals ρ on M satisfying $\rho(1) = \|\rho\| = 1$, where 1 denotes the unit of M.

1.6.7 The relationship between JB-algebras and JBW-algebras, mimicing that of C*-algebras and W*-algebras, is a powerful tool. Its strength is particularly evident when any JB-algebra is viewed, via the canonical injection, as a subset of its second dual. This is demonstrated by the next theorem.

Theorem 1.6.8 ([Sh1] [HaSt, 4.4.3, 4.7.5])

Let A be a JB-algebra. Then

(a) A^{**} is a JBW-algebra;

(b) the product of A^{**} extends the usual product on A and is separately weak* continuous;

(c) the weak continuous extension of each state of A to A^{**} is normal;*

(d) A^{**} is the monotone completion of A, i.e. A^{**} is the smallest monotone closed subalgebra of A^{**} containing A;

(e) if A is a JC-algebra then A^{**} is a JW-algebra;

(f) if A is a JBW-algebra and a JC-algebra then it is a JW-algebra.

In this manner it is common to regard a JB-algebra A as a subalgebra of A^{**} and to identify states of A with normal states of its bidual.

1.7 Projections

1.7.1 Let M be a JBW-algebra. The idempotents of M are called *projections*. Given an element a in M let $W(a)$ denote the weak* closure of $C(a)$. Then $W(a)$ is an abelian JBW-algebra with a unit, denoted by $r(a)$ and which is called the *range projection* of a in M [AlShSt] [HaSt, 4.1.10, 4.2.6]. The JBW-algebra $W(a)$, is isometrically isomorphic to the self adjoint part of a W*-algebra. Thus, M is amply endowed with projections. Through the order structure inherited from M, the set of all projections in M is a complete lattice and the set of all central projections is a complete sublattice. For each projection e of M, $c(e)$ denotes the least central projection of M majorising e, termed the *central support* of M.

1.7.2 If e is a projection in the JBW-algebra M then $U_e(M)$ is a JBW-subalgebra of M with unit e. Moreover, as e ranges over all projections, the $U_e(M)$ are precisely the weak* closed quadratic ideals of M [Ed, 2.3]. The weak* closed ideals of M are of the form $z \pm M$, where z is a central projection [HaSt, 4.3.6]. In particular, $c(e) \pm M$ is the weak* closed ideal of M generated by a projection e.

A non-zero projection e of M is said to be *minimal* if it does not majorise any other non-zero projection. Thus, e is a minimal projection in M if and only if $U_e(M) = \mathbb{R}e$. M is said to be a *factor* if $Z(M) = \mathbb{R}1$. Thus, M is a factor if and only if it has no non-trivial central projections.

1.7.3 Let ρ be a normal state on a JBW-algebra M. The *support* of ρ, denoted by $s(\rho)$, is defined to be the least projection e of M such that $\rho(e) = 1$, whereas the *central support* of ρ, $c(\rho)$, is the least central projection z such that $\rho(z) = 1$. We have $c(\rho) = c(s(\rho))$.
When A is a JB-algebra and ρ is a state of A, and therefore a normal state of A^{**}, by $s(\rho)$ and $c(\rho)$ we mean the support and central support, respectively, in A^{**}.

1.7.4 Using [AlSh1, 1.13], [AlSh2] and [AlShSt], the map $\rho \mapsto s(\rho)$ describes a bijection from the set of pure normal states of a JBW-algebra M to the set of minimal projections of M. The inverse map is given for a minimal projection e, by $e \mapsto \rho_e$, where $\rho_e(x)e = \{exe\}$ for all x in M. In terms of a JB-algebra A, this translates to a bijection between the set of pure states of A and set of minimal projections of A^{**}.

1.8 Type Decomposition of a JBW-Algebra

1.8.1 The notion of type classification in JB-algebras and JBW-algebras is, in essence, a translation of the respective concepts for C*-algebras and W*-algebras. As a type I W*-algebra is not necessarily type I as a C*-algebra, it is the convention to use the term postliminal to refer to those C*-algebras originally classed as type I. In consequence, the concept of postliminality is also used for JB-algebras. For details regarding postliminal JB-algebras see section (1.11).
Our particular interest is the type decomposition of JBW-algebras (actually JBW*-algebras), a characterisation in terms of abelian projections, mimicing that for W*-algebras. Such a decomposition will be invaluable in our later work.

Recall that a projection e in a JBW-algebra M is said to be *abelian* if $U_e(M)$ is an abelian JBW-algebra.

1.8.2 Let M be a JBW-algebra. Then M is said to be

(i) *type I* if there exists an abelian projection p in M with $c(p) = 1$;

(ii) *continuous* if M has no non-zero abelian projections.

Furthermore, M is said to be *type* I_n, where $n < \infty$, if there exist n abelian projections $p_1, \ldots, p_n$ in M with $c(p_1) = \ldots = c(p_n) = 1$ and $\sum p_i = 1$. We define M to be *type I finite* if it is an ℓ^∞-sum of such type I_n JBW-algebras. If M is type I without type I finite part then it is said to be of type I_∞.

1.8.3 Let e be a minimal projection in a JBW-algebra M. Then $c(e) \circ M$, the weak* closed ideal generated by e, is a type I factor, N say. The norm closed ideal in N, and hence in M, generated by its minimal projections is called the *elementary ideal* of N and is denoted by $K(N)$. Further details of type I factors are provided in sections (1.12) and (1.13).

1.8.4 Let p and q be projections in JBW-algebra M. Then p and q are said to be *exchanged by a symmetry*, denoted by $p \sim_1 q$, if $p = U_s(q)$ for some symmetry s in M.

We note the following technical result, of use in Chapters Three and Four.

Lemma 1.8.5 ([HaSt, 5.3.2])

Let M be a JBW-algebra with projections p and q. Then

(a) if p is abelian and $q \le p$ then $q = c(q)p$;

(b) if p and q are abelian with $c(p) = c(q)$ then $p \sim_1 q$;

(c) if M is type I then there exists an abelian projection r in M such that $r \le q$ and $c(q) = c(r)$.

In particular, it follows from (a) that in a JBW-factor every abelian projection is minimal.

1.8.6 Projections p and q in a JBW-algebra are said to be *equivalent*, denoted by p » q, if there exists a finite sequence $s_1; :::; s_n$ of symmetries such that $q = U_{s_1} ::: U_{s_n}(p)$. If p » q then $c(p) = c(q)$.

Theorem 1.8.7 ([Top1, Theorem 9])

Let M be a type I finite JBW-algebra. Then

(a) *if* (e_i)*and* (f_i)

1.10 Exceptional Decomposition

A JB-algebra A is said to be *purely exceptional* if every factor representation is onto the exceptional JB-algebra N_3^8.

Proposition 1.11.3 ([Bu2, 3.9])

Let A be a JB-algebra. Then there exists a Jordan ideal J of A such that

(a) *J is postliminal;*

(b) *A/J is antiliminal.*

1.11.4 A *composition series* of JB-algebra A is a strictly increasing family of closed ideals (I_λ), indexed by an ordinal segment $[0, \alpha]$, such that

(i) $I_0 = 0$ and $I_\alpha = A$;

(ii) for each limit ordinal γ

$$I_\gamma = \overline{\bigcup_{\lambda<\gamma} I_\lambda};$$

where bar denotes norm closure;

(iii) each successive quotient is of the form $I_{\lambda+1}=I_\lambda$.

This definition is a similar to that for C*-algebras, with Jordan ideals replacing ideals. We note the following result, which is a Jordan analogue to that in [Kap] [Ped1, 6.2.6].

Theorem 1.11.5 ([Bu1, 2.3.2])

Let A be a JB-algebra. Then the following are e-23eQivalent.

1.12 Spin Factors

Lemma 1.12.1 ([HaSt, 6.1.3])

Let H be a real Hilbert space of dimension at least two. Let $A = H \oplus \mathbb{R}1$ with norm and product defined by:

$$
\begin{aligned}
\|a + \lambda 1\| &= \|a\|_2 + |\lambda| \\
(a + \lambda 1) \circ (b + \mu 1) &= (\mu a + \lambda b) + (\langle a, b\rangle + \lambda\mu)1 \quad \text{for all } a, b \in H \;\; \lambda, \mu \in \mathbb{R}.
\end{aligned}
$$

Then A is a JW-algebra factor.

1.12.2 The JW-algebras that occur in the above lemma are the

1.13 The Universal Enveloping C*-Algebra

1.13.1 Throughout the thesis, by an *involution* on a C*-algebra we shall always mean a *-antiautomorphism of order two. Let B be a C*-algebra with involution ®. We will habitually use the notation $B^{®}$ to denote the set of points of B fixed by ®.

The following structure is crucial to our later results, particularly to those in Chapters Three and Four. For further details refer to [HaSt, *x*7]. Throughout this section let A be a JC-algebra.

1.13.2 A *universal enveloping C*-algebra* of A is a pair ($C^{¤}(A); \tilde{A}$), where $C^{¤}(A)$ is a C*-algebra and $\tilde{A}$ is an injective Jordan homomorphism

$$\tilde{A} : A \;¡\; ! \; C^{¤}(A)_{sa}$$

such that

(i) $C^{¤}(A)$ is the C*-algebra generated by $\tilde{A}(A)$;

(ii) A has the *universal extension property*. That is, each Jordan homomorphism ¼ : $\tilde{A}(A)$! C_{sa}, where C is any C*-algebra, extends uniquely to a *-homomorphism $\tilde{¼}$: $C^{¤}(A)$! C.

1.13.3

Proposition 1.13.6 ([Han2, §2])

(a) *Let A be a JC-algebra. Then the following are equivalent.*

  (i) *A is universally reversible.*

  (ii) *$A = C^*(A)^{\alpha}_{sa}$.*

  (iii) *A has no spin factor representations other than those onto U_2 or U_3.*

(b) *Let M be a JW-algebra. Then the following are equivalent.*

  (i) *M is universally reversible.*

  (ii) *$M = W^*(M)^{\alpha}_{sa}$.*

  (iii) *The type I_2 part of M is isomorphic to $C(X; U_2) \oplus C(Y; U_3)$, where X and Y are compact hyperstonean spaces.*

1.13.7 Let H be a complex Hilbert space and let $v : H \to H$ be a conjugate linear isometry. Define an involution $\alpha : B(H) \to B(H)$ by $x \mapsto vx^*v^*$. Evidently α is of order two if and only if $v^2 = \lambda 1$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Since v commutes with v^2 we see that v commutes with λ. In which case, as v is conjugate linear, λ is real. It follows that $v^2 = \pm 1$.

Such an involution α is said to be a *real flip* if $v^2 = 1$, or a *quaternionic flip* if $v^2 = -1$. A conjugate linear isometry v is called a *conjugation* if $v^2 = 1$ and a *unit quaternion* if $v^2 = -1$.

Œ1.95Tf8.912.95TD[TJ/F411.95Tf2311.9710.7208TD

Suppose that α is a real flip. Then $H^v = \{h \in H)$ by

On the other hand, suppose that α is a quaternionic flip. Let $k = iv$ so that $\{1, i, v, k\}$ generate the quaternions. To form a quaternionic Hilbert space $H_{\mathbf{H}}$ define a quaternionic inner product, denoted by $< .,. >_{\mathbf{H}}$,

$$< a, b >_{\mathbf{H}} = Re < a, b > - iRe < ia, b > - vRe < va, b > - kRe < ka, b >$$

for all $a, b \in H$. It is apparent that any element x in $B(H)$ satisfies $\alpha(x) = x^*$ if and only if $vx = xv$ and $kx = xk$. Thus elements of $B(H)^{\alpha}_{sa}$ are self adjoint $\mathbf{H}-$linear operators, that is, $B(H)^{\alpha}_{sa} \cong B(H_{\mathbf{H}})_{sa}$.

This discussion forms the basis of the next theorem.

Theorem 1.13.8 ([AlHaSh, 3.1])

Let M be a universally reversible JW-algebra factor of type I_n $(2 \leq n \leq \infty)$. Then there exists a complex Hilbert space H such that M is isomorphic to one of the following.

(a) $B(H)^{\alpha}_{sa} \cong B(H_{\mathbf{R}})_{sa}$, *where α is a real flip on $B(H)$ and $H_{\mathbf{R}}$ is a real Hilbert space.*

(b) $B(H)_{sa}$.

(c) $B(H)^{\beta}_{sa} \cong B(H_{\mathbf{H}})_{sa}$, *where β is a quaternionic flip on $B(H)$ and $H_{\mathbf{H}}$ is a quaternionic Hilbert space.*

1.13.9 A non-abelian type I JW-algebra factor M is said to be *real, complex* or *quaternionic* if M is isomorphic to $B(K)_{sa}$ where K is a real, complex

1.14 JB*-Algebras and JC*-Algebras

Finally we describe the algebraic complexification of JB-algebras and JC-algebras, namely JB*-algebras and JC*-algebras.

1.14.1 A *JB*-algebra*, originally termed a *Jordan C*-algebra*, is a complex Banach space A which is a complex Jordan algebra with involution * and that satisfies the following conditions.

(JB1) $kx \pm yk \cdot kxkkyk$ for all $x; y\ 2\ A$.

(JB2) $kx^{¤}k = kxk$ for all $x\ 2\ A$.

(JB3) $kU_x(x^{¤})k = kxk^3$ for all $x\ 2\ A$.

Chapter 2

Preliminaries of Jordan Triple Systems

2.1 Introduction

Taking a similar role to Chapter One, this chapter presents the principal background to Jordan triple systems. Material has been selected from a wide variety of literature, perhaps the most consistently used of which are [FrRu4] and [FrRu5], with the intention to support later work by providing relevant information in a clear and concise manner.

We start by describing general Jordan triple systems, before moving on to define JB*-triples. We naturally include a summary of well-known results pertaining to the focus of the thesis, that is the inner ideals of JB*-triples. In particular, we present carefully chosen results from the wide ranging work of Edwards and Ruttimann [EdRü1]-[EdRü8]. We also establish some technical lemmas regarding inner ideals, which do not warrant deep examination, but which are required in subsequent chapters.

The notion of type decomposition, specifically that provided by Horn [Ho2], dominates later work, and is therefore given a thorough exposition here. We go on to give details of atomic decomposition, and of Friedman and Russo's Gelfand Naimark Theorem for JB*-triples. To conclude, we supply a succinct presentation of the Stone-Weierstrass Theorem and its many generalisations.

2.2 Jordan *-Triple Systems over $\mathbb{C}$

2.2.1 A *Jordan *-triple system over* $\mathbb{C}$ is a complex vector space A with a triple product $A \times A \times A \to A$, denoted by $(a, b, c) \mapsto \{abc\}$, such that

(i) $\{...\}$ is linear in a and c, conjugate linear in b;

(ii) $\{abc\} = \{cba\}$ for all $a, b, c \in A$;

(iii) $\{ab\{xyz\}\} = \{\{abx\}yz\} + \{xy\{abz\}\} - \{x\{bay\}z\}$.

The identity (iii) is often known as the *main identity*.

Henceforth, by a Jordan *-triple system we shall mean a Jordan *-triple system over $\mathbb{C}$.

2.2.2 Let A be a Jordan triple system. Then a subspace B of A is said to be a *subtriple* if $\{BBB\} \subset B$. A subspace I of A is said to be an *ideal* if $\{AAI\} + \{AIA\} \subset I$, and an *inner ideal* if $\{IAI\} \subset I$.

2.2.3 For a pair of elements x and y in a Jordan triple A define a linear operator on A by $D(x, y)(z) = \{xyz\}$, and a conjugate linear operator on A by $Q(x)(z) = \{xzx\}$. Using the commutator notation, $[X, Y] = XY - YX$, we can rewrite the main identity (iii) as follows.

(iii)$'$ $[D(a, b), D(x, y)] = D(\{abx\}, y) - D(x, \{bay\})$.

2.2.4 We state a series of well-known identities satisfied by a Jordan triple system. (P1-P3) are referred to as *polarisation identities*.

(I) $fybfxaxgg = 2ffybxgaxg$ ¡ $fxfbyagxg$.

(II) $fxafxbxgg = fxfaxbgxg = ffxaxgbxg$.

(III) $fyafxaxgg = fyfaxagxg$.

(IV) $ffxaxgbfxaxgg = fxfafxbxgagxg$.

(V) $ffxaxgafxaygg = fxaffxaxgaygg$.

(VI) $2fyafxazgg = fyfaxagzg + fyfazagxg$.

(P1) $4fxyzg = \sum_0^3 i^k fx + i^k y \quad x + i^k y \quad zg$.

(P2) $4fzxzg = \sum_0^3 (¡1)^k fx + i^k z \quad x + i^k z \quad x + i^k zg$.

(P3) $2fxyzg = fx + z \quad y \quad x + zg$ ¡ $fxyxg$ ¡ $fzyzg$.

2.2.5 Let A be a Jordan *-triple system. An element e in A is said to be a *tripotent* if $e = feeeg$. For such a tripotent e, let $D = D(e;e)$ and $Q = Q(e)$. We have th-349(tr75(ha)26(v)27(e)-32661)]TJ/F311.9GTf11.750T6g1

The associated *Peirce projections* P_0, P_1 and P_2 are defined as follows.

$$\begin{aligned} P_2 &= Q^2 = D(2D - I) \\ P_1 &= 2(D - Q^2) = 4D(I - D) \\ P_0 &= I - 2D + Q^2 = (I - D)(I - 2D). \end{aligned}$$

The Peirce projections are mutually orthogonal linear projections on A with $P_0 + P_1 + P_2 = I$. Thus, $A = P_0(A) \oplus P_1(A) \oplus P_2(A)$. Furthermore, since

$$(D - I)P_2 = (D - \frac{I}{2})P_1 = (D - 0)P_0 = 0,$$

it follows that $P_i(A) = ker(D - \frac{iI}{2})$, for $i = 0, 1, 2$. That is, for each $i \in \{0, 1, 2\}$ the Peirce i-space is the $\frac{i}{2}$-eigenspace of D.

2.2.6 Let A be a Jordan *-triple system and let $A_i = P_i(A)$, for $i = 0, 1, 2$. The following rules, often called *Peirce rules*, hold.

(i) $\{A_i A_j A_k\} \subseteq A_{i-j+k}$ if $i - j + k \in \{0, 1, 2\}$.

(ii) $\{A_i A_j A_k\} = 0$ if $i - j + k \notin \{0, 1, 2\}$.

(iii) $\{A A_2 A_0\} = \{A A_0 A_2\} = 0$.

Each A_i (i=0, 1, 2) is a subtriple of A, and A_0 and A_2 are inner ideals of A. When it is necessary to emphasise which tripotent e is being considered we will use the notations $P_i = P_i^e$ and $A_i = A_i(e)$.

2.3 Jordan *-Triple Systems and Jordan * Algebras

2.3.1 Let A be a Jordan *-triple system and let $a \in A$. Then we can define an algebra, denoted by $A^{(a)}$ and called the *a-homotope of A*, by defining a product $x \circ y = \{xay\}$, for all x and y in A. Using the symmetric property of the triple product and identity (V), we have that, for all $x, y \in A$,

$$\begin{aligned} x \circ y &= y \circ x; \\ x^2 \circ (x \circ y) = \{\{xax\}a\{xay\}\} &= \{xa\{\{xax\}ay\}\} = x \circ (x^2 \circ y). \end{aligned}$$

Hence $A^{(a)}$ is a complex Jordan algebra.

More specifically, the following proposition demonstrates that if we take a tripotent e in A then the triple system is locally a complex Jordan * algebra, in the sense that the Peirce 2-space $A_2(e)$ is a complex Jordan * algebra.

Proposition 2.3.2

*Let A be a Jordan *-triple system. Let e be a tripotent in A. Then $A_2(e)$ is a complex Jordan * algebra with identity e, and with product and involution given by: $x \circ y = \{xey\}$, $x^\# = \{exe\}$.*

On the other hand, every complex Jordan * algebra is a Jordan triple system with triple product given by

$$\{xyz\} = (x \circ y^*) \circ z + x \circ (y^* \circ z) - (x \circ z) \circ y^*. \tag{2.1}$$

2.4 Ordering the Tripotents of a Triple System

2.4.1 Let e and f be two tripotents of a Jordan triple system A. Then e and f are said to be *orthogonal*, denoted e ? f

2.5 JB*-Triples and JBW*-Triples

We now introduce the structure within which our work is set, that is, the JB*-triple, a category which includes C*-algebras and JB*-algebras.

2.5.1 A *JB*-triple* is a complex Banach space A which is a Jordan *-triple system such that for all a in A

(i) $kfaaagk = kak^3$;

(ii) D(a,a) is an hermitian operator on A with non-negative spectrum.

Naturally every norm closed subtriple of a JB*-triple is itself a JB*-triple.

2.5.2 A *JBW*-triple* is a JB*-triple M with a Banach space predual, denoted by M¤. This predual is necessarily unique and the triple product on M is separately ¾(M,M¤) continuous [BaTi, 2.1]. It follows that a weak* closed

By polarisation, it follows that if $\pi : A \longrightarrow B$ is a linear map such that $\pi(\{xxx\}) = \{\pi(x)\pi(x)\pi(x)\}$ for all x in A, then π is a triple homomorphism. Triple isomorphisms between JB*-triples correspond precisely to surjective linear isometries.

Theorem 2.6.2 ([Kaup, 5.5][Ho1, 2.4])

Let A and B be JB-triples and let $\pi : A \longrightarrow B$. Then π is a surjective linear isometry if and only if π is a triple isomorphism.*

2.8 Tripotents in JB*-Triples

2.8.1 Let e be a non-zero tripotent in a JB*-triple A. Then e is said to be:-

(i) *minimal* if $A_2(e) = \mathrm{C}e$;

(ii) *complete*, if $A_0(e) = f0g$;

(iii) *unitary* if $A_2(e) = A$;

(iv) *abelian* if $A_2(e)$ is an associative JB*-algebra (hence, an abelian C*-algebra).

Theorem 2.8.5 ([FrRu4, Proposition 4])

Let M be a JBW-triple. There is a bijective correspondence between the elements of $\partial_e(M_{*,1})$ and the minimal tripotents of M given by $\rho \mapsto s(\rho)$*

2.9.4 One result of the preceding discussion is that if M is a JBW*-triple with $x \in M$, and $\overline{M_x}$ denotes the JBW*-subtriple of M generated by x, then $\overline{M_x}$ is an abelian W*-subalgebra of $M_2(r(x))$. In particular, $x \in M_2(r(x))_+$. Since there exists a complete tripotent in M majorising $r(x)$, [Ho1, 3.12], we have the following.

Lemma 2.9.5

If $x \in M$, where M is a JBW-triple, then $x \in M_2(u)_+$ for some complete tripotent u of M.*

2.10 Norm Closed Inner Ideals, Ideals and Quotients

Crucially, JB*-triples are stable under appropriate quotients.

Theorem 2.10.1 ([Kaup, p523][FrRu5, p146])

Let A be a JB-triple with norm closed ideal J. In the quotient norm A/J is (a) a JB*-triple; (b) a JC*-triple if A is a JC*-triple.*

We remark that (2.10.1(b)) anticipates the Gelfand-Naimark Theorem discussed in (2.12).

2.10.2 Norm closed inner ideals and norm closed ideals in JB*-triples have geometric characterisations. In the following, (a) was obtained in [EdRü4] and (b) in [BaTi].

Theorem 2.10.3

Let A be a JB-triple.*

(a) *A JB*-subtriple I of A is an inner ideal of A if and only if each $\rho \in I^*$ has unique norm preserving extension in A^*.*

(b) *A norm closed subspace J of A is an ideal of A if and only if J is an M-ideal of A.*

2.10.4 Let A be a JB*-triple. Elements a and b in A satisfy $D(a; b) = 0$ if and only if $D(b; a) = 0$ [EdRü6, 3.1]. In which case, a and b are said to be *orthogonal*, written $a \perp b$. For any subset B of A, the annihilator $B^{\perp}$ of B in A, which is defined to be the set

$$B^{\perp} = \{a \in A : a \perp b \text{ for all } b \in B\};$$

is a norm closed inner ideal of A [EdRü6, 3.2]; it is weak* closed if A is a JBW*-triple.

2.10.5 For norm closed ideals in JB*-triples, the defining algebraic condition can be relaxed. In fact, (see [BuCh], [DinTi], [Har2]), a norm closed subspace I of A is an ideal of A if and only if it satisfies any one of the following equivalent conditions:

$$(i) \quad \{AAI\} \subseteq I \quad ; \quad (ii) \quad \{AIA\} \subseteq I \quad ; \quad (iii) \quad \{AII\} \subseteq I.$$

2.10.6 Let I and J be norm closed ideals of a JB*-triple A. By definition, $\{IAI^{\perp}\} \subseteq I \cap I^{\perp} = \{0\}$. Via this remark, the previous paragraph and the fundamental identity, we see that

$$\{AI\{AI^{\perp}I^{\perp}\}\} \subseteq \{\{AIA\}I^{\perp}I^{\perp}\} + \{A\{IA2I$$

I and J are said to be *orthogonal* if $D(a;b) = 0$ for all $a \in I$ and $b \in J$. By functional calculus, if $x \in I \cap J$ then $x = \{yyy\}$ for some $y \in I \cap J$. It follows that $\{AI$ J

In the above context, $J^{\perp}$ is often referred to as the *complementary* ideal of the weak* closed ideal J of M. Also, we shall see, (2.10.21), that part (c) generalises to any weak* closed ideal in a JBW*-triple.

2.10.9 Let A be a JB*-triple, let J be a norm closed ideal of A and let I be a norm closed inner ideal of A. Every minimal tripotent of the weak* closed inner ideal I^{**} of A^{**} is again minimal in A^{**}. The corresponding statement is true for the weak* closed ideal J^{**} of A^{**}. On the other hand, by (2.10.8(d)), a minimal tripotent of A^{**} lies in J^{**} or in its complementary ideal. These remarks, together with (2.8.5), have the following consequences.

(a) Each $\rho \in \partial_e(I_1^*)$ has a unique extension in $\partial_e(A_1^*)$.

(b) Each of the following maps is a bijection.

(i) $\{\rho \in \partial_e(A_1^*) : s(\rho) \in I^{**}\} \to \partial_e(I_1^*) \quad (\rho \mapsto \rho_{|_I})$.

(ii) $\{\rho \in \partial_e(A_1^*) : \rho(J) \neq 0\} \to \partial_e(J_1^*) \quad (\rho \mapsto \rho_{|_J})$.

(iii) $\{\rho \in \partial_e(A_1^*) : \rho(J) = 0\} \to \partial_e((A/J)_1^*) \quad (\rho \mapsto \tilde{\rho})$, where $\tilde{\rho}$ denotes the functional given by $\tilde{\rho}(x + J) = \rho(x)$, whenever ρ lies in the stated domain.

When confusion seems unlikely we will tend to identify the respective domain and codomain in the correspondences (i),(ii), and (iii). It is in this sense that we write

$$\partial_e(A_1^*) = \bigcup \partial_e(I_1^*),$$

where the union ranges over all norm closed inner ideals I of A.

2.10.10 Let B be a subtriple of a JBW*-triple M. Then B is said to be *complemented* in M if $M = B \oplus \mathrm{Ker} B$ [LoNe], where

$$\mathrm{Ker} B = \{a \in M : \{BaB\} = 0\}.$$

A projection $P : M \to M$ is said to be a *structural* projection if

$$P\{aP(b)a\} = \{P(a)bP(a)\}$$

for all $a, b \in M$.

These notions of complementation and structural projections were introduced into JBW*-triples, and JB*-triples, by Edwards and Rüttimann [EdRü7] with significant effect.

Theorem 2.10.11 ([EdRü7, 4.5, 4.8] [EdMcRü, 5.5, 5.6])
Let M be a JBW-triple. Then*

(a) *all structural projections on M are contractive and weak* continuous;*

(b) *a subtriple of M is complemented if and only if it is a weak* closed inner ideal;*

(c) *the map, $P \mapsto P(M)$, is a bijection from the set of structural projections of M onto the set of weak* closed inner ideals of M.*

If J is a weak* closed ideal of a JBW*-triple M, so that $M =$

2.10.12 Edwards' and Rüttimann's description of the weak* closed inner ideals of a W*-algebra will prove to be vital. We let $CP(W)$ denote the set of pairs of centrally equivalent projections of a W*-algebra W.

Theorem 2.10.13 ([EdRü1, 4.1])

Let W be a W-algebra. Then the map $(e; f) \mapsto eWf$, is an order preserving bijection from $CP(W)$ onto the set of weak* closed inner ideals of W.*

The norm closed inner ideals of a C*-algebra are precisely the intersections of closed left and right ideals [EdRü3, 2.6]. A similar result holds the weak* closed inner ideals of W*-algebras [EdRü1, 3.16].

2.10.14 We end this section with a few technical lemmas regarding triple ideals and inner ideals that are required in subsequent chapters.

Lemma 2.10.15

Let M be a JBW-triple. Let I be a weak* closed inner ideal of M and let J be a weak* closed ideal of M. Let $P : M \longrightarrow J$ be the natural projection. Then*

(a) $I = I \cap J \oplus I \cap J^{\perp}$;

(b) $I \cap J^{\perp} = (I \cap J)^{\perp} \cap I$;

(c) $P(I) = I \cap J$.

Proof

(a) Let $x \in I$. Then, by functional calculus, $x = \{yyy\}$ for some $y \in I$. Using the ideal decomposition of M, there exists $a \in J$ and $b \in J^{\perp}$ such that $y = a + b$. Thus

$$x = \{yyy\} = \{yay\} + \{yby\} \in I \cap J \oplus I \cap J^{\perp}.$$

The converse is clear.

(b) Since $I \cap J$ is a weak* closed ideal of I we see that $I = I \cap J \oplus (I \cap J)^{\perp} \cap I$. Comparison with (a) gives the result.

(c) As P is the identity on J and vanishes on $J^{\perp}$ we have

$$P(I \cap J) = I \cap J \quad \text{and} \quad P(I \cap J^{\perp}) = 0.$$

The conclusion now follows via part (a). $\square$

Lemma 2.10.16

Let M be a JBW-triple. Let (J_α) be a family of weak* closed ideals of M such that M is the ℓ^{∞}-sum, $\sum J_\alpha$. If I is a weak* closed inner ideal of M then $I =$*

Proof

By the main identity

$$
\begin{aligned}
fJAKg &= fJAfKKKgg = ffJAKgKKg + fKKfJAKgg \;¡\; fKfAJKgKg \\
&= 2ffJAKgKKg ½ fIKKg ½ K:
\end{aligned}
$$

Similarly $fJAKg ½ J$ so that $fJAKg ½ J \backslash K = 0$. Finally, by (2.10.20), $T(J) \backslash T(K) = 0$ and so $T(J) ? T(K)$.

The remaining statement follows from the separate weak* continuity of the triple product. *2*

2.11 Types of JBW*-Triples

2.11.1 The type classification of a JBW*-triple is a natural, though not obvious, analogue of that of a JBW*-algebra. Details of the latter can be found in [HaSt]. The study of type I JBW*-triples was initiated in [Ho1] and subsequently pursued in considerable detail in [Ho2] and [Ho3]. The structure of continuous JBW*-triples was investigated to resolution in [HoNe].

2.11.2 Let M be a JBW*-triple. Then M is said to be a *type I JBW*-triple* if every weak* closed ideal of M contains an abelian tripotent. Equivalently, M is type I if it contains a complete tripotent e such that $M_2(e)$ is a type I JBW*-algebra [Ho1, 4.14]. M is said to be *type* I_1 if and only if it contains no29TD[(non-zero8(1)(t.95T767.10TD[(typ)4glete)-36k*)-.62-23186TD[(if)-297(ev)66aid toe6]TJ/F1

A finer type II and type III classification of continuous JBW*-triples is also given in [HoNe].

2.11.3 A JBW*-triple M is said to be a *factor* if it contains no non-trivial weak* closed ideals. The type I factors are precisely those that contain a minimal tripotent and, by [Ho2, 1.8], they are the *Cartan factors*, briefly described in the following.

It what follows we let H and K be complex Hilbert spaces of respective orthonormal dimensions n and m, where n and m are, possibly infinite, cardinals. Let $j : H \longrightarrow H$ be a conjugation.

(a) Rectangular, $R_{n,m}$: $M = B(H, K)$. If $n \leq m$, realising H as a closed subspace of K, let p be the orthogonal projection onto H. Then

$$M = B(pK, K) \cong B(K)p$$

and, taking any involution, $\psi : B(K) \longrightarrow B(K)$, we also have that $M \cong \psi(p)B(K) \cong B(K, \psi(p)K) \cong B(K, H)$. Thus $R_{m,n} \cong R_{n,m}$, and the rectangular Cartan factors are the weak* closed left (or right) ideals of type I von Neumann factors. In the special case of $n = 1$, then $M = K$ and the triple product can be realised as follows.

$$\{xyz\} = \frac{1}{2}\left(< x, y > z + < z, y > x \right)$$

If $1 \leq n, m < \infty$, then $M = M_{n,m}(\mathbb{C})$.

(b) Hermitian, $S_n(\mathbb{C})$: $M = \{x \in B(H) : x = jx^*j\}$.

The map $\alpha : B(H) \longrightarrow B(H)$ given by $x \mapsto jx^*j$ is a real flip, (see (1.13.7)), giving $M = B(H)^{\alpha}$, a JW*-algebra factor of type I. Moreover, if $1 \leq n < \infty$ then M is isomorphic to the $n \times n$ symmetric matrices (hence, the notation $S_n(\mathbb{C})$).

(c) Symplectic, $A_n(\mathbb{C})$ $(2 \le n \le \infty)$: $M = \{x \in B(H) : x = -jx^*j\}$.

If n is even and finite, or is infinite, then there is a unit quaternion $v : B(H) \to B(H)$, (see(1.13.7)), and the induced map β given by $\beta(x) = -vx^*v$, ($\beta : B(H) \to B(H)$), is a quaternionic flip. In which case, $M \cong B(H)^{\beta}$, via $x \mapsto -vjx$, again a JW*-algebra factor of type I. When $2 \le n < \infty$, M is identified with the $n \times n$ antisymmetric matrices.

(d) Complex spin factors: $M = V_\lambda = U_\lambda \oplus iU_\lambda$, the complexification of the real spin factor U_λ defined in (1.12). Henceforth, we refer to the V_λ as *the* spin factors and the U_λ as the *real spin factors*.

The previous four kinds of Cartan factors are JC*-triples with the following overlappings: $M_{1,3}(\mathbb{C}) \cong A_3(\mathbb{C})$, $S_2(\mathbb{C}) \cong V_2$, $M_2(\mathbb{C}) \cong V_3$, $A_4(\mathbb{C}) \cong V_5$ and $M_1(\mathbb{C}) \cong S_1(\mathbb{C}) \cong A_2$.

As ρ ranges over $\partial_e(M_{*,1})$, let M_{at} denote the ℓ^∞-sum of the distinct C_ρ's that arise. Then M_{at} is the smallest weak* closed ideal of M containing all minimal tripotents of M, whereas $M_{at}^\perp$ contains no minimal tripotents. M_{at} is called the

2.12 Atomic and Cartan Factor Representations

2.12.1 Let A be a JB*-triple. By a *Cartan factor representation* of A we mean a (triple) homomorphism, $\pi : A \to M$, where M is a Cartan factor and $\pi(A)$ is weak* dense in M. The *rank* of such a Cartan factor representation is defined to be the rank of M.

2.12.2 Given $\rho \in \partial_e(A_1^*)$, where A is a JB*-triple, let C_ρ be the weak* closed Cartan factor ideal of A^{**} generated by $s(\rho)$, (see (2.11.6)), and let $P_\rho : A^{**} \to C_\rho$ be the natural projection. Let $\pi_\rho : A \to C_\rho$

Correspondingly, the restriction to A is the map

$$\pi_{at} = \bigoplus \pi_\rho : A \to \Big(\bigoplus C_\rho\Big)_\infty \qquad (a \mapsto \bigoplus \pi_\rho(a));$$

called the *atomic representation* of A. If $a \in A$ with $\pi_{at}(a) = 0$ then, by these remarks, $\tau(a) = 0$ for all $\tau \in \partial_e(A$

2.13 The Stone-Weierstrass Theorem

2.13.1 The Stone-Weierstrass Theorem, a generalisation of the Weierstrass approximation Theorem, was proved by M.H. Stone in 1937. It is the driving force behind much of the thesis. We state a non-unital version.

Theorem 2.13.2 (Stone-Weierstrass)

Let X be a locally compact Hausdorff space and let A be a closed subalgebra of $C_0(X)$ such that

(a) for each x in X there exists $f \in A$ such that $f(x) \neq 0$;

(b) A separates the points of X;

(c) if $f \in A$ then $\overline{f} \in A$.

Then $A = C_0(X)$.

2.13.3 In the context of C*-algebras a more elegant statement is possible. Let A be a commutative C*-algebra. Then the Gelfand map is an isometric *-isomorphism of A onto $C_0(P(A))$, where $P(A)$ is given the weak* topology. Conversely, via the usual evaluation map, every locally compact Hausdorff space X can be identified with $P(C_0(X))$. This leads to the following reformulation of (2.13.2).

Theorem 2.13.4

Let A be a commutative C-algebra. Let B be a C*-subalgebra of A such that B separates the points of $P(A) \cup \{0\}$. Then $B = A$.*

2.13.5 At the time of writing, the extension of (2.13.4) to all C*-algebras, the *Stone-Weierstrass Conjecture*, remains an open problem. Progress has been made towards generalising (2.13.4), for instance, by weakening the constraint of commutativity, (Kaplansky successfully proved the result for postiliminal

C*-algebras [Kap]), or by enlarging of the set of functionals under consideration.

2.13.6 As JB*-triples and JB*-algebras are generalisations of C*-algebras, it is natural to consider a meaningful version of the Stone-Weierstrass Theorem for these structures. Due to the lack of positivity and thus of pure states in JB*-triples, pure states are replaced by the extreme points of the dual ball. Subtriples take the role given to subalgebras. In this way, the JB*-triple version of the Stone-Weierstrass conjecture is as follows.

Conjecture 2.13.7

Let A be a JB-triple and let I be a subtriple of A such that I separates $\partial_e(A_1^*) \cup \{0\}$. Then $A = I$.*

The JB*-triple counterpart of (2.13.4) has been shown to hold [FrRu3, 3.4]. Sheppard has obtained extensions of the Stone-Weierstrass Theorem, in particular, to postliminal JB-algebras and postliminal JB*-triples [Shep2, 4.11], [Shep3, 5.5]. We also note the following, which we use later.

An attempt is then made to describe the weak* inner ideals of particular JW*-triples, namely universally reversible JW*-algebras. Our motivation is the role these structures occupy in many of the arguments of the next chapter. Initially, our intention was to extend [EdRüVa2] to formulate a general resolution, that is, to prove that every weak* inner ideal of a universally reversible JW*-algebra M has a unique representation in the form $eM\phi(e)$, where e is a projection in $W^*(M)$ with ϕ-invariant central support. This proved not to be possible, the symplectic part forming an obstacle. So the restriction is made that M has no non-zero symplectic part. This is sufficient for our subsequent needs.

Finally, an account of Cartan factor representation theory is offered, as this represents a vital tool in what follows.

3.2 The Centroid of a JB*-Triple

3.2.1 Let A be a JB*-triple. The *centroid* of A is the set of $T \in B(A)$ satisfying $T(\{a\ b\ c\}) = \{Ta\ b\ c\}$ for all $a, b, c \in A$, and will be denoted by $C_e(A)$. Equivalently, for $T \in B(A)$, the condition that $T \in C_e(A)$ is characterised, separately, by each of the following conditions.

$$(i) T(\{a\,b\,c\}) = \{a\,b\,Tc\}; \ (ii) TD(a,b) = D(a,b)T; \ (iii) TD(a,a) = D(a,a)T.$$

The centroid of a JB*-triple was introduced and studied in [DinTi] and developed further in [EdRü9] and [EdLoRü].

3.2.2

Proof

(a) We have

Proof (of (a))

Let $T, S \in C_e(M)$. The restriction of T to the inner ideal $M_2(u)$ lies in the centroid of the JBW*-algebra $M_2(u)$, so that $T(u) \in Z(M_2(u))$ by the result of [DinTi] mentioned in (3.2.4). Furthermore,

$$ST(u) = ST\{u\,u\,u\} = \{S(u)\;u\;T(u)\} = S(u) \circ T(u);$$

and $T^{\#}(u) = T^{\#}\{u\,u\,u\} = \{u\;T(u)\;u\} = (T(u))^{*}$. Therefore, ψ is a *-homomorphism.

Proof

Let $S \in C_e(M)$. Then, by (3.2.6), there exists a unique $a \in Z(M_2(u))$ such that $S(u) = a$. Since $P_0^u = I - 2D(u;u) + P_2^u = 0$, for each $x \in M$ we have $x = 2\{uux\} - P_2^u(x)$, so that

$$S(x) = 2\{S(u)\,u\,x\} - S(P_2^u(x)) = 2D(a;u)(x) - D(a;u)P_2^u(x)$$ we)we,a-1.63-7.3TD[(2)]TJ

3.2.9 Consider a JBW*-triple $M = A\bar{-}C$, where A is an abelian von Neumann algebra and C is a Cartan factor. Let u be a complete tripotent of C. Then $1 - u$ is a complete tripotent of M and we have

$$M_2(1 - u) = A\bar{-}C_2(u):$$

Since $C_2(u)$ is a type I JBW-algebra factor, it follows that

$$Z(M_2(1 - u)) = A - u:$$

Let $T\ 2\ C_e(M$

3.3 The Type of an Inner Ideal

3.3.1 In this section, in tensor product notation of the form $A \bar{\otimes} C$, the left hand side will always represent an abelian von Neumann algebra. Recall Horn's type I structure theorem [Ho2, 1.7].

Theorem 3.3.2
If M is a type I JBW-triple then $M \cong \left(\sum A_i \bar{\otimes} C_i\right)_\infty$, for (up to isomorphism) distinct Cartan factors C_i. Moreover such a decomposition is unique.*

3.3.3 The uniqueness above is not stated in [Ho2] but is implicit and can be seen as follows. Suppose there is a surjective linear isometry

$$\pi : \left(\sum A_i \bar{\otimes} C_i\right)_\infty \to \left(\sum B_j \bar{\otimes} D_j\right)_\infty ,$$

where the C_i are distinct (up to isomorphism) Cartan factors, and similarly for the D_j. Fix i_0. By (3.2.10(b)) there exist projections $z_j \in B_j$ such that $\pi(A_{i_0} \bar{\otimes} C_{i_0}) = \sum z_j B_j \bar{\otimes} D_j$. Pick a j_0 such that $z_{j_0} \neq 0$. Thus, again by (3.2.10(b)), there is a non-zero weak* closed ideal J of A_{i_0} such that $\pi(J \bar{\otimes} C_{i_0}) = z_{j_0} B_{j_0} \bar{\otimes} D_{j_0}$, so that $C_{i_0} \cong D_{j_0}$, by [Ho3, §4]. By the uniqueness of the D_j's, this implies that $z_j = 0$ for all $j \neq j_0$. Thus,

$$\pi(A_{i_0} \bar{\otimes} C_{i_0}) = z_{j_0} B_{j_0} \bar{\otimes} D_{j_0},$$

giving, $\left(\sum_{i \neq i_0} A_i \bar{\otimes} C_i\right)_\infty \cong \left(\sum_{j \neq j_0} B_j \bar{\otimes} D_j\right)_\infty + (1 - z_{j_0}) B_{j_0} \bar{\otimes} D_{j_0}$. Now, the above argument, starting with the j_0-summand on the right, contradicts the uniqueness of the C_i if $z_{j_o} \neq 1$. That is, there is a unique j_0 such that

$$\pi(A_{i_0} \bar{\otimes} C_{i_0}) = B_{j_0} \bar{\otimes} D_{j_0},$$

in which case $A_{i_0} \cong B_{j_0}$. By tak

3.3.4 We remark that up to (Jordan) *-isomorphism there is one and only one

(a) exceptional JBW*-algebra factor;

(b) spin factor of given dimension ¸ (possibly infinite);

(c) type I_n JW*-algebra factor of given Cartan type and rank n, where n

Otherwise, C and D are JW*-algebras and π_2 extends to a *-isomorphism $\tilde{\pi}_2 : W^*(C) \to W^*(D)$ (of W*-algebras). This induces a *-isomorphism between von Neumann algebras [KaRi2],

$$\pi_1 \bar{\otimes} \tilde{\pi}_2 : A \bar{\otimes} W^*(C) \to B \bar{\otimes} W^*(D);$$

which, by restriction, sends $A \bar{\otimes} C$ onto $B \bar{\otimes} D$.

The general case is now immediate from the homogeneous decomposition of type I JBW*-algebras together with (3.3.3). $\Box$

Lemma 3.3.6

Let M be a JBW-triple with complete tripotent u. Suppose that, for ideals I_i of $M_2(u)$, $M_2(u)$ is the ℓ^∞-sum $\sum I_i$. Put $u = \sum u_i$, where $u_i \in I_i$, for each i and let $J_i = T^w(I_i)$, for each i. Then*

(a) $M = \left(\sum J_i\right)_\infty$;

(b) $M_2(u) = \sum (J_i)_2(u_i)$ *and each u_i is a complete tripotent of J_i.*

Proof

(a) This is immediate from (2.10.8(c)).

(b) For each i, u_i is the identity element of I_i (regarded as a JBW*-algebra) and $(J_i)_2(u_i) = (I_i)_2(u_i) = I_i$, by (2.10.8(b)). $\Box$

: V_n = 3J/F5 7.2(with

3.3.7 From this point on we concentrate on JW*-triples of type I. Let M be a JW*-triple with complete tripotent u such that $M_2(u) \cong A \bar{\otimes} C$, where C is a (special) Cartan factor; thus C is isomorphic to a type I JW*-algebra factor. As C varies, the possiblities for M, up to isomorphism, are as set out in the table below. Here n represents a cardinal number, possibly infinite, and we mean ℓ^{∞}-sum. We make tacit use of (3.2.10(c)) throughout.

	C	M
[Ho2; 4.1]	V_n ($n \neq 3, 5$)	$A \bar{\otimes} V_n$
[Ho2; 5.5]	$R_{n,n}$	$\sum A_m \bar{\otimes} R$ V_n

Since the complete tripotent u was unspecified, it follows from the table that $M_2(u) \cong M_2(v)$ for any other complete tripotent v of M. Since $1 \otimes w$ is a complete tripotent of $A \bar{\otimes} C$, whenever w is a complete tripotent of C, we therefore have $M_2(u) \cong A \bar{\otimes} C_2(w)$, for every complete tripotent w of C.

Proposition 3.3.10

Let M be a type I JW-triple. Let u and v be complete tripotents of M. Then $M_2(u) \cong M_2(v)$.*

Proof

We have $M = \left(\sum J_i\right)_\infty$, where each $J_i \cong A_i \otimes C_i$ for a certain abelian von Neumann algebras A_i and Cartan factors C_i. We also have $u = \sum u_i$ and $v = \sum v_i$, where u_i and v_i are complete tripotents of J_i, for each i. By (3.3.9), $(J_i)_2(u_i) \cong (J_i)_2(v_i)$, for each i. Therefore,

$$M_2(u) = \sum (J_i)_2(u_i) \cong \sum (J_i)_2(v_i) = M_2(v). \qquad \Box$$

3.3.11 Let M be a Cartan factor with a tripotent u. Then we say that u is of *rank n* if $M_2(u)$ is of rank n. Let C be an hermitian, a rectangular or symplectic Cartan factor of rank at least two. Let u be a complete tripotent of C of rank two. The structure of $C_2(u)$ relative to the type of C is as follows.

C : hermitian; rectangular; symplectic.

$C_2(u)$:

Lemma 3.3.12

Let M be a JW-triple such that $M = A \overline{-} C$, where A is an abelian von*

3.3.13 We now show that the "generic type" of a type I JW*-triple (mostly) determines, and is determined by, that of a weak* closed inner ideal.

Theorem 3.3.14

Let M be a JW-triple such that $M \cong A \bar{\otimes} C$, where A is an abelian von Neumann algebra and C is a Cartan factor of rank at least two. Let I be a weak* closed inner ideal of M such that I has no type I_1 part. Then M is hermitian (respectively rectangular, symplectic) if and only if I is hermitian (respectively rectangular, symplectic).*

Proof

We have $I = \left(\sum I_i\right)_\infty$, where, for each i, $I_i \cong A_i \bar{\otimes} C_i$, with each A_i an abelian von Neumann algebra and C_i a Cartan factor of rank at least two. For each i choose a tripotent $u_i \in I_i$ such that $(I_i)_2(u_i)$ is type I_2. We have that $M_2(u_i) = (I_i)_2(u_i)$ is type I_2 for each i. By definition I is hermitian if and only if C is hermitian. Thus, the above equality, together with (3.3.12(a)), implies that M is hermitian if and only if I is hermitian. The remaining claims, those in parenthesis, are obtained in the same way using (3.3.12(b) and (c)). $\square$

Corollary 3.3.15

Let I be a weak closed inner ideal in a type I JW*-triple M such that neither I nor M have type I_1 part. If M is hermitian (respectively rectangular, symplectic) then I is hermitian (respectively rectangular, symplectic). The converse is true if I generates M as a weak* closed ideal.*

Proof

Now, M is the ℓ^∞-sum $\sum J_i$, where, for each i,

If M is hermitian then each of the J_i is hermitian. In which case, each non-zero $J_i \setminus I$ is hermitian by (3.3.14), implying that I is hermitian.
Conversely, suppose that I is hermitian and generates M as a weak* closed ideal. By (2.10.16), the latter condition implies that $J_i \setminus I$ is non-zero for all i. Now (3.3.14) implies that each J_i, and therefore M, is hermitian.
Directly similar arguments, through (3.3.14), give the remaining cases. 2

3.3.16 We now turn to type I_1 JW*-triples, first recalling some particular features of type I rectangular JW*-triples.

(a) Let W be a von Neumann algebra with partial isometry u and let $l = uu^{¤}$ and $r = u^{¤}u$ so that $W_2(u) = lWr$. We have $lu = u = ur$ and

Proposition 3.3.17

The following are equivalent for a JBW-triple M.*

(a) *M is type I_1.*

(b) *$M \cong We$, for some abelian projection e*

To conclude this section, we present the following two results, of use in the next chapter.

Proposition 3.3.19

Let M be a JBW-algebra with complete tripotent u such that $M_2(u)$ is isomorphic to a W*-algebra of type I. Then M is isomorphic to a W*-algebra of type I.*

Proof

By definition (see [Ho2, *x*5]) M is a type I rectangular JBW*-algebra. So, $M \cong We$ for some type I W*-algebra W and projection e in W (3.3.16(c)). Since M is a JBW*-algebra, We has a unitary tripotent v, giving

$$We = vv^{*}Wev^{*}v = vv^{*}Wv^{*}v \;»$$

Lemma 3.3.20

3.4.4 Given a universally reversible JW*-algebra M and a projection $e \in M$, the canonical involution ϕ of $W^*(M)$ restricts to an involution on $eW^*(M)e$, with $(eW^*(M)e)^{\phi} = eMe$. Thus, it is necessary, generally, to consider involutions other than the possible canonical ones. It is convenient to make the following definition. A JW*-algebra will said to be *complex* if it is *-isomorphic to a von Neumann algebra.

3.4.5 Let α be an involution on a von Neumann algebra W. Then α is said to be *central* if it fixes each point of $Z(W)$, and is said to be *split* if $1 = z + \alpha(z)$ for some non-trivial central projection z in $Z(W)$. The properties of α are intimately connected to those of the JW*-algebra of α-fixed points, $W^{\alpha} = \{x \in W : \alpha(x) = x\}$. One elementary observation is that if e is a projection of W^{α}, then α is an involution on eWe with $(eWe)^{\alpha} = eW^{\alpha}e$. Another is as follows.

Lemma 3.4.6

Let α be a central involution on a von Neumann algebra W and let e be a projection in W^{α}. Then α is central on eWe.

Proof

This is immediate, since $Z(eWe) = eZ(W) = eZ(W^{\alpha}) = Z(eW^{\alpha}e)$. $\square$

Now, as an involution α on a von Neumann algebra W restricts to an involution on $Z(W)$, the following is immediate from [HaSt, 7.3.4 , 7.3.5] and (3.4.6).

Lemma 3.4.7

Let α be an involution on a von Neumann algebra W. Then α is either central, split or there is a non-trivial projection $z \in Z(W^{\alpha})$ such that α is central on Wz and split on $W(1 - z)$.

Lemma 3.4.8

Let ® be an involution on a von Neumann algebra W and let e be a projection in W. Then ®($c(e)$) = c(®(e)).

Proof

Since e · $c(e)$, ®(e) · ®($c(e)$) and so c(®(e)) · ®($c(e)$). Through this principle, $c(e)$ = c(®(®(e))) · ®(c(®(e))), giving ®($c(e)$) · c(®(e)) and hence equality. *2*

We next state two key results of Gåsemyr [Gå]. The first is [Gå, 2.2(a), 2.8].

Theorem 3.4.9

Let ® be an involution on a von Neumann algebra W. Then

*(a) the type I_2 part of $W^®$ is *-isomorphic to (A_1 - V_2)©(A_2 - V_3)©(A_3 - V_5), where A_1, A_2 and A_3 are abelian von Neumann algebras;*

(b) if $W^®$ has no abelian part, then it generates W.

3.4.10 As a JW*-algebra is universally reversible precisely if its type I_2 part is *-isomorphic to (A_1 - V_2) © (A_2 - V_3), where A_1 and A_2 are abelian von Neumann algebras (1.13.6), the next proposition is immediate from [Gå, 2.2(b)].

Proposition 3.4.11

*Let ® be an involution on a von Neumann algebra W such that $W^®$ is universally reversible without abelian part. Then there is a *-isomorphism ° : W ! $W^¤(W^®)$ such that ® = $°^{¡1}$Á°, where Á is the canonical involution on $W^®$.*

The next two results are complementary.

Lemma 3.4.12

Let α be an involution on a von Neumann algebra W. We have

(a) *if α is split then W^α is complex;*

(b) *if W^α is complex and has no abelian part then α is split.*

Proof

(a) Let z be a non-trivial projection in $Z(W)$ such that $\alpha(z) = 1 - z$, and consider the map $\pi : Wz \to W^\alpha$ given by $\pi(x) = x + \alpha(x)$. Since Wz and $\alpha(Wz) = (1 - z)W$ are orthogonal, a straightforward check shows that π is an injective Jordan *-homomorphism. Moreover, given $a \in W^\alpha$ we have that $a = az + a(1 - z) = \pi(az)$. So π is surjective.

(b) Suppose that the stated conditions hold. By [HaSt, 7.4.7] the canonical involution ϕ on W^α is split. Also, (3.4.11) implies that $\alpha = \gamma^{-1}\phi\gamma$ for some *-isomorphism $\gamma : W \to W^*(W^\alpha)$. Therefore, α is split. $\square$

Lemma 3.4.13

Let α be an involution on a von Neumann algebra W. We have

(a) *α is not central if and only if there exists a non-trivial projection z in $Z(W)$ such that $z\alpha(z) = 0$;*

(b) *if W^α has no complex part, then α is central;*

(c) *if α is central and W^α is complex, then W^α is abelian;*

(d) *if W^α has no abelian part and α is central, then W^α has no complex part.*

Proof

(a) If α is not central then by (3.4.7) there is a non-trival projection p in $Z(W^{\alpha})$ such that α is split on Wp, from which the required conclusion follows by definition. The converse is immediate from the definition.

(b) If α is not central, we can choose a non-trivial projection $z \in Z(W)$ such that $z\alpha(z) = 0$ by part (a). Then $Wz \cong (z + \alpha(z))W^{\alpha}$, by the proof (3.4.12(a)).

(c) Since α is invariant on the complement of the abelian part of W, this is immediate from (3.4.12(b)).

(d) Suppose that $W^{\alpha}z$ is complex for some unique projection z in W^{α}. If α is central and W^{α} has no abelian part then, applying (3.4.12(b)) to α on Wz, we conclude that $z = 0$. $\square$

3.4.14 Let A be an abelian von Neumann algebra. Consider $M = A \bar{\otimes} B(H)^{\alpha}$, where α is a real quaternionic flip and where $dim(H) \geq 3$. The von Neumann algebra generated by M is $W = A \bar{\otimes} B(H)$, and $\phi = id \otimes \alpha$ is an involution on W such that $M = W^{\phi}$. By (3.4.11), we may suppose that $W = W^{*}(M)$ and that ϕ is the canonical involution. If α is a real flip the dimension condition can be relaxed to $dim(H) \geq 2$.

Let e be an abelian projection in M. Then $c(e) \in Z(M) = A \otimes 1$, so that $c(e) = z \otimes 1$, for some $z \in A$. Let f be a minimal projection in $B(H)^{\alpha}$. Then $z \otimes f$ is abelian in M and $c(z \otimes f) = z \otimes 1 = c(e)$. Therefore, by (1.8.5(b)), there is a symmetry s in M such that $ses = z \otimes f$.

(a) If α is a real flip, then f is minimal in $B(H)$ [Shep1, 3.2.3(i)]. Therefore, $e = s(z \otimes f)s$ is abelian in $W^{*}(M)$. Since ϕ is a central involution, it follows that $eW^{*}(M)e = Z(eW^{*}(M)e) = Z(eMe) = eMe$.

(b) Let ® be a quaternionic flip. Then $f = p + Á(p)$, for some minimal projection p in $B(H)$ [Shep1, 3.2.3(iii)]. Thus, with $q = z - p$, we have that q is abelian in $W^{¤}(M)$ and also that $ses = q + Á(q)$.

Summing over homogeneous parts, we conclude:

Lemma 3.4.15

Let M be a universally reversible type I JW-algebra and let e be an abelian projection in M. We have*

(a) if M is hermitian then e is abelian in $W^{¤}(M)$ and $eMe = eW^{¤}(M)e$;

(b) if M is symplectic then $e = p + Á(p)$, where p is an abelian projection of $W^{¤}(M)$ and Á is the canonical involution.

Proposition 3.4.16

Let M be a universally reversible JW-algebra with no complex part. Let Á be the canonical involution on $W^{¤}(M)$. Suppose that M has non-zero symplectic type I part. Then there exists a non-zero projection e in $W^{¤}(M)$ such that*

(a) e is abelian in $W^{¤}(M)$;

(b) $e + Á(e)$ is abelian in M;

(c) $eMÁ(e) = 0$.

Proof

Let N be the symplectic type I part of M. By (3.4.15(b)) we can choose a non-zero projection e in $W^{¤}(N)$ such that e is abelian in $W^{¤}(N)$ and $e + Á(e)$ is abelian in N. Then, since $W^{¤}(N)$ is an ideal of $W^{¤}(M)$, this projection e is abelian in $W^{¤}(M)$ and $e + Á(e)$ is abelian in M.

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To a□rm part (c) we observe that, by (3.4.13(b)), Á is central and so

$$\begin{aligned}(e + Á(e))M(e + Á(e)) &= Z^{\text{i}}(e + Á(e))M(e + Á(e))^{¢} = (e + Á(e))Z(M)\\ &= (e + Á(e))Z(W^{¤}(M)):\end{aligned}$$

Therefore, as $eÁ(e) = 0$,

$$eMÁ(e) = e^{\text{i}}(e + Á(e))M(e + Á(e))^{¢}Á(e) = e(e + Á(e))Á(e)Z(W^{¤}(M)) = 0:2$$

Proposition 3.4.17

Let M be a universally reversible JW-algebra with no complex part. Let e be a non-zero projection in* $W^{¤}(M)$ *such that* $eMÁ(e) = 0$. *Then*

(a) $eW^{¤}(M)e \cong (e + Á(e))M(e + Á(e))$;

(b) *e is abelian in* $W^{¤}(M)$ *and* $e + Á(e)$ *is abelian in* M;

(c) $Mc(e)$ *is type I symplectic;*

where Á is the canonical involution on

Secondly, we claim that $N^{Á} \cong eN$. This follows, through the proof of (3.4.12(a)), after we note that

$$e(exe + Á(e)yÁ(e)) = exe = (exe + Á(e)yÁ(e))e:$$

Now, through the two preceding statements, we can confirm that

$$eW^{¤}(M)e = eN \cong N$$

Let $f = e + Á(e)$. Clearly fz is an abelian projection in $Mc(e)$. However, fz cannot be abelian in $W^{*}(M)c(e)$. Indeed, since $zfW^{*}(M)f$ is a weak* closed ideal of $fW^{*}(M)f$, which is a type I_2 W*-algebra, and as

$$zfW^{*}(M)f = fzW^{*}(M)c(e)fz;$$

it follows that $fzW^{*}(M)c(e)fz$ is of type I_2. Thus, in the light of (3.4.15(a)), the projection fz provides a contradiction and so the hermitian part of $Mc(e)$ is zero. Finally as, by assumption, M has no complex part, $Mc(e)$ must be symplectic. *2*

Now (3.4.16) together with (3.4.17) gives the following.

Theorem 3.4.18

Let M be a universally reversible JW-algebra with no complex part. Then there exists a non-zero projection e in $W^{*}(M)$ such that $eMÁ(e) = 0$ if and only if M has non-zero symplectic type I part.*

Such a projection e satisfies the following, where Á is the canonical involution.

(a) $eW^{*}(M)e \cong (e + Á(e))M(e + Á(e))$.

(b) $(e + Á(e))W^{*}(M)(e + Á(e)) \cong A - M_2(\mathbf{C})$, *for some abelian von Neumann algebra A.*

(c) $Mc(e)$ *is symplectic type I.*

We note the following corollary.

Corollary 3.4.19

Let M be any JW-algebra without symplectic part or complex part. Then every complete tripotent of M is unitary.*

Proof

Let u be a complete tripotent of M. Then

$$(1 - uu^*)M(1 - u^*u) = M_0(u) = \{0\}.$$

Put $e = (1 - uu^*)$. Then $\psi(e) = 1 - u^*u$, where ψ is the canonical involution on $W^*(M)$, and hence $eM\psi(e) = 0$. Now, by hypothesis and (3.4.18), $e = 0$ and thus $\psi(e) = 0$. So, $uu^* = 1 = u^*u$. $\square$

Lemma 3.4.20

Let u be an abelian tripotent in an hermitian JW-algebra M. Then u is an abelian tripotent of $W^*(M)$ and $M_2(u) = W^*(M)_2(u)$.*

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(M) , and 2
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ab11.95T,f1530 2TD[(W)]Tgeneou0TD[(us)]T4cenot]T4ceco49(o/F475795Tf20.

Proof

We may suppose that M

that $v = ev + v\psi(e)$, giving $f = ef + v\psi(e)v^*$. In particular, $ef = fe$. Let $p = (1 - e)f$. Since, by assumption, $M_2(v) \subseteq M_1(u)$, we deduce that

$$pM\psi(p) = (1 - e)(fM\psi(f))\psi(1 - e) \subseteq (1 - e)(M_1(u))\psi(1 - e) = \{0\}.$$

Therefore, because (3.4.18) now shows that $p = 0$, $f = ef$. Similarly, $e = fe = f$. Hence, $0 = v\psi(e)v^* = v\psi(f)v^* = vv^* = f$. Thus, $v = 0$ and likewise $u = 0$; a contradiction. $\square$

Proposition 3.4.23

Let I be a weak closed inner ideal of an hermitian JW*-algebra M. Then there is a unitary tripotent u of I such that $I = M_2(u)$. Putting $e = uu^*$ (in $W^*(M)$), we have $I = eM\psi(e)$, where ψ is the canonical involution of $W^*(M)$. Moreover, if I is type I_1 then it is abelian.*

Proof

Pick a complete tripotent u of I. If I has no type I_1 part then the first part of the statement is immediate from (3.3.15) and (3.4.19).

On the other hand, suppose that I is type I_1. If I is not abelian then, passing to a weak* closed ideal, in order to obtain a contradiction we may suppose that $I \cong A \otimes H$, where A is an abelian von Neumann algebra and H is a Hilbert space of dimension at least two. Choose orthonormal elements h_1 and h_2 of H. Then h_1 and h_2 are rigidly collinear. Now, since $P_i^{(1-h_j)} = id - P_i^{h_j}$, for $(i; j = 1; 2)$, $1 - h_1$ and $1 - h_2$ are rigidly collinear. Thus, the tripotents of I that correspond to $1 - h_1$ and $1 - h_2$ are rigidly collinear tripotents of M, contradicting (3.4.22). $\square$

We have reached the following extension of (3.4.2).

Theorem 3.4.24

Let M be a universally reversible JW-algebra with no non-zero symplectic part. Then the map, e 7! $eMÁ(e)$, is an order preserving bijection from $P(W^{¤}(M);Á)$ onto the set of weak* closed inner ideals of M, where $Á$ is the canonical involution of*

We will also require the next proposition.

Proposition 3.4.25

Let M be a universally reversible JW-algebra without type I_1 part. Let I be a weak* closed rectangular inner ideal of M without type I_1 part. Then $I = eMÁ(e)$, for some projection e in $W^{¤}(M)$, where $Á$ is the canonical involution of $W^{¤}(M)$.*

Proof

Let Mz be the weak* closed ideal of M generated by I, where z is a central projection in M. The involution $Á$ of $W^{¤}(M)$, through restriction, gives rise to the canonical involution of $W^{¤}(Mz) = W^{¤}(M)z$. Further, since M is universally reversible, $Mz = W^{¤}(M)^{Á}z$ [HaSt, 7.3.3]. Now, through (3.3.20), Mz is *-isomorphic to a type I W*-algebra. Thus, via (3.4.2), there exists a projection e in $W^{¤}(M)z$ such that $I =$

3.5 Cartan Factor Representation Theory

3.5.1 Cartan factor representation theory is a key technique used within the thesis, and thus warrants a clear account. The material given in this section, the majority of which is known, is influenced by that contained in [BuChZa1]. Simplistically, the Cartan factor representation structure of a JB*-triple A is (mostly) determined by the structure of its second dual. Specifically, if C is a finite dimensional Cartan factor, we shall show here that A has a Cartan factor representation onto C precisely when the bidual of A has a weak* closed ideal that is isomorphic to $C(X) \otimes C$, where X is some compact hyperstonean space.

Proposition 3.5.2

Let C be a Cartan factor and let A be a JB-triple. Then there exists a Cartan factor representation $\pi : A \longrightarrow C$ if and only if C is isometric to a weak* closed ideal of A^{**}_{at}.*

Proof

Suppose that such a Cartan factor representation $\pi : A \longrightarrow C$ exists. Then $A^{**} = C \oplus ker\tilde{\pi}$, where $\tilde{\pi} : A^{**} \longrightarrow C$ is the weak* continuous extension of π. Clearly $C \subset A^{**}_{at}$ and since $ker\tilde{\pi}$ is a weak* closed ideal so is C [Ho1, §4]. Conversely suppose that C is a weak* closed ideal of A^{**}_{at} with natural projection $P : A^{**} \longrightarrow C$. Then the restriction of P to A is a Cartan factor representation of A. $\square$

Proposition 3.5.3

Let C be a finite dimensional Cartan factor and X be a compact Hausdorff space. Let D be a Cartan factor such that $D \subset C(X) \otimes C$. Then D is isometric to a subfactor of C.

Proof

For each x in X define $\pi_x : C(X) \otimes C \to C$ to be the linear map satisfying $\pi_x(f \otimes a) = f(x)a$. Then π_x is a Cartan factor representation and moreover $\{\pi_x : x \in X\}$ is a faithful family of Cartan factor representations. The faithfulness condition follows from the fact that $C(X) \otimes C \cong C(X; C)$, via $f \otimes a \mapsto f(.)a$; if $b = \sum_1^n f_i \otimes a_i \neq 0$ then $g = \sum_i^n f_i(.)a_i \neq 0$ so that, for some $x \in X$, $\pi_x(b) = g(x) \neq 0$.

Suppose that D is non-zero and choose x in X such that $\pi_x(K(D)) \neq 0$. Such an x exists since the family is faithful, and since $K(D)$ is, by definition, non-zero. As $K(D)$ is simple we have $K(D) \cong \pi_x(K(D)) \subset C$. It follows that $K(D)$ is finite dimensional and is thus reflexive, so that

$$K(D) = K(D)^{**} = D;$$

and so D is isometric to a subfactor of C, as required. $\square$

Proposition 3.5.4 ([BuChZa1, 2.3])

Let C be a finite dimensional Cartan factor and let A be a JB-triple. Then all Cartan factor representations of A are onto C if and only if $A^{**} = C(X) \otimes C$ for some compact Haudsdorff space X.*

3.5.5 A JB*-triple A is said to be *type* C, for some finite dimensional Cartan factor C, if $A/\ker\pi \cong C$, for all Cartan factor representations π of A, that is, if all Cartan factor representations of A are onto C. By convention, the zero triple is of every type.

We remark that by the previous theorem, if a JB*-triple A is type C, where C is a finite dimensional Cartan factor, then the bidual of A is isomorphic to $C(X) \otimes C$, for Tf9.24.16(theor96)]TJ(Ato)]Tn619i96A

Theorem 3.5.6 ([BuChZa2, 5.1])

Let A be a JB-triple with a Cartan factor representation with rank n, where $n < \infty$. Then either all Cartan factor representations of A have rank at most n, or A contains a non-zero ideal J such that*

(a) all Cartan factor representations of J have rank greater than n;

(b) all Cartan factor representations of A/J have rank at most n.

We have the following extension of (3.5.4).

Proposition 3.5.7

Let C be a finite dimensional Cartan factor and let A be a JB-triple. Then there exists a Cartan factor representation $\pi : A \to C$ if and only if there exists a weak* closed ideal J of A^{**} such that $J \cong C(X) \otimes C$, for some compact hyperstonean space X.*

Proof

Let $\pi : A \to C$ be a Cartan factor representation. Then, via (3.5.2), C is isometric to a weak* closed ideal of A^{**}_{at} so that the result follows.

Conversely, suppose that there exists a weak* closed ideal J of A^{**} such that $J \cong C(X) \otimes C$. Let $P : A^{**} \to J$ be the natural weak* continuous projection. Then $J = \overline{P(A)}$, where bar denotes weak* closure. Since

$$P(A) \cong A/(A \cap \ker P);$$

which is a quotient of A, it is enough to show that $P(A)$ has a Cartan factor representation onto C.

As $P(A) \subseteq J$, we have that $P(A)^{**} \subseteq J^{**} \cong C(Y) \otimes C$, for some compact hyperstonean space Y. Let D be a weak* closed Cartan factor ideal of $P(A)^{**}$. Then, by (3.5.3), D is isometric to a Cartan subfactor of C.

It is now immediate from (3.5.2) that all Cartan factor repesentations of $P(A)$ are onto subfactors of C.

Let $C_0, C_1, \dots, C_n$ be the Cartan factors arising from the Cartan factor representations of $P(A)$. Then, using (3.5.6), there is a composition series of norm closed ideals of $P(A)$, $(J_i)_{0 \le i \le n+1}$, with J_{i+1}/J_i homogenous type C_i, for distinct Cartan factors C_i. Thus, via (3.5.4), as remarked in (3.5.5),

$$P(A)^{**} \cong \sum \left(J_{i+1}/J_i \right)^{**} \cong \sum C(X_i) \otimes C_i,$$

where both sums are ℓ^∞-sums.

Finally, there is a weak* continuous triple homomorphism

$$\Psi : P(A)^{**} \longrightarrow \overline{P(A)} = J \cong C(X) \otimes C,$$

and hence $P(A)^{**}$ has a weak* closed ideal isomorphic to $C(X) \otimes C$. Therefore $C_i = C$ for some i, and consequently $P(A)$ has a Cartan factor representation onto C. $\Box$

3.5.8 We make the following aside. Let $J \cong C(X) \otimes C$ be a weak* closed ideal of the bidual of some JB*-triple, where C is a finite dimensional Cartan factor and X is some compact hyperstonean space. Let B be a JB*-triple contained in J. Using the argument contained in the proof of the previous proposition, (3.5.7), we see that every Cartan factor representation of B is necessarily onto a subfactor of C. We can go further and deduce that $B^{**} \cong \left(\sum C(X_i) \otimes C_i \right)_\infty$, for some compact hyperstonean spaces X_i, where the C_i denote these subfactors of C.

The next two propositions are counterparts of (3.5.4) and (3.5.7).

Proposition 3.5.9

Let A be a JB-triple. Then the following are equivalent.*

(a) A^{**} is type I_1.

(b) All Cartan factor representations of A are onto Hilbert spaces.

Proof

(a) $\Rightarrow$ (b) Assume (a) and let $\pi : A \longrightarrow C$ be a Cartan factor representation of A. Then C is isometric to a weak* closed ideal J of A^{**} (3.5.2). Let u be a non-zero tripotent in J. Choose a tripotent v of A^{**} such that u is a projection in $A_2^{**}(v)$. Since $A_2^{**}(v)$ is an abelian JW*-algebra (by (3.3.17(c))), and because $J_2(u)$ is a subfactor of it, we must have $J_2(u) = \mathbb{C}u$ so that u is a minimal tripotent of J. Thus all tripotents of J are minimal and hence J is a Hilbert space [DaFr, p308]. It is now evident that C is a Hilbert space.

(b) $\Rightarrow$ (a) Assume (b). Then the atomic part, M, of A^{**} is an ℓ^{∞}-sum of Hilbert spaces and so is type I_1 [Ho2, §2]. However, A embeds as a JB*-subtriple of M (2.12.3). So, A^{**} can be realised as a JBW*-subtriple of M^{**}, and it follows from (3.3.18) that A^{**} is type I_1. $\Box$

Proposition 3.5.10

The following are equivalent for a JB-triple A.*

(a) A has a Cartan factor representation onto a Hilbert space.

(b) A^{**} has non-zero type I_1 part.

Proof

(a) $\Rightarrow$ (b) This is clear.

(b) $\Rightarrow$ (a) Assume (b) and let $P : A^{**} \to J$ be the natural projection, where J denotes the type I_1 part of A^{**}. Then $P(A)^{**}$ is contained in J^{**} so that $P(A)^{**}$ is type I_1 by (3.3.18). Hence, all Cartan factor representations of $P(A)$ are onto Hilbert spaces (3.5.9). However, $P(A)$ is a quotient of A, giving (a). $\square$

Chapter 4

The Inner Stone-Weierstrass Theorem for Universally Reversible JC*-Algebras

4.1 Introduction

The ultimate aim of the thesis is to determine inner ideals in JB*-triples by extreme points of their dual balls. The means by which this objective is achieved is what we, from now on, choose to term the *Inner Stone-Weierstrass Theorem*:

Let A be a JB-triple with norm closed inner ideals I and J, such that $I \subseteq J$. Suppose that $\partial_e(I_1^*) = \partial_e(J_1^*)$. Then $I = J$.*

In this chapter, we establish the Inner Stone-Weierstrass Theorem for universally reversible JC*-algebras, a theorem which is exploited in Chapter Five to prove the full theorem for JB*-triples.

Let I and J be norm closed inner ideals in a universally reversible JC*-algebra A, such that I ½ J. To observe that I is equal to J, it is enough to prove the equality of the corresponding biduals in $A^{¤¤}$. This JW*-algebra decomposes into a continuous part and a type I part, and the latter part in turn decomposes into summands of the form $B \overline{-} C$, where C is a Cartan factor and B is an abelian von Neumann algebra [Ho2]. Making use of the Cartan factor representation theory of section (3.5), we find that the assumption that $I_{at}^{¤¤} = J$

4.2 Inner Ideals and the Atomic Part

4.2.1 This section takes the form of a series of necessary technical lemmas investigating the atomic part of the bidual of a norm closed inner ideal. Let A be a JB*-triple with norm closed inner ideal I. We shall prove that when $\partial_e(I_1^*) = \partial_e(A_1^*)$, the biduals of A and I have the same atomic part, or alternatively, that I separates $\partial_e(A_1^*) \cup \{0\}$, thereby validating our terminology. Furthermore, we will demonstrate that when $A_{at}^{**} = I_{at}^{**}$, we can conclude that A is equal to I, if either A is a C*-algebra or I is a triple ideal. Finally, utilising the exposition of Chapter Three (3.5), we will show that given this equality of atomic parts, I has a Cartan factor representation onto a specfic Cartan factor C precisely when A does.

Lemma 4.2.2

Let M be an atomic JBW-triple with weak* closed inner ideal I. Then I is atomic.*

Proof

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Lemma 4.2.3

Let A be a JB-triple and let I be a norm closed inner ideal of A.*

Let $\rho \in \partial_e(A_1^*)$. Then

(a) $I_{at}^{**} = I^{**} \cap A_{at}^{**}$;

(b) $\rho(I) = 0$ *if and only if* $\rho(I_{at}^{**}) = 0$.

Proof

(a) Clearly $I_{at}^{**} \subseteq I^{**} \cap A_{at}^{**}$. Conversely, since $I^{**} \cap A_{at}^{**}$ is a weak* closed inner ideal of A_{at}^{**}, it follows by (4.2.2) that it is atomic. However it is also clearly a weak* closed inner ideal of I^{**} and so $I^{**} \cap A_{at}^{**} \subseteq I_{at}^{**}$.

(b) Let $\rho \in \partial_e(A_1^*)$ such that $\rho(I) = 0$. Then as $\rho(I_{at}^{**})$ is contained in $\rho(I^{**})$ and the latter is zero by weak* continuity of ρ, we see that $\rho(I_{at}^{**}) = 0$. Conversely suppose that $\rho(I_{at}^{**}) = 0$. By part (a) $I_{at}^{**} = I^{**} \cap A_{at}^{**}$, and so, via (2.10.15(a)), we have

$$\begin{aligned} I^{**} &= I^{**} \cap A_{at}^{**} \oplus I^{**} \cap (A_{at}^{**})^{\perp} \\ &= I_{at}^{**} \oplus I^{**} \cap (A_{at}^{**})^{\perp}. \end{aligned}$$

Therefore $\rho(I^{**}) = 0$, as required. $\square$

4.2.4 It is now possible to give the following equivalent conditions, that demonstrate, amongst other things, that the coincidence of the atomic parts of the biduals of a JB*-triple and its inner ideal can be represented in terms of a Stone-Weierstrass separation condition.

Theorem 4.2.5

Let A be a JB-triple and let I be a norm closed inner ideal of A. Then the following are equivalent.*

(a) $\rho(I) \neq 0$ *for all* $\rho \in \partial_e(A^*_1)$.

(b) $I^{**}_{at} = A^{**}_{at}$.

(c) *For all* $\rho \in \partial_e(A^*_1)$ *the restriction of* ρ *to* I *lies in* $\partial_e(I^*_1)$.

(d) *The unique extension map from* $\partial_e(I^*_1)$ *to* $\partial_e(A^*_1)$

(e)) (a) Suppose that there exists ½ 2 $@_e(A_1^{¤})$ such that ½(I) = 0. Then ½ agrees with the zero function on I. It follows that I does not separate $@_e(A_1^{¤})$ [$f0g$. 2

4.2.6 The maximal norm closed left and right ideals of a C*-algebra are completely classified in terms of pure states.

Theorem 4.2.7 ([Ped1, 3.13.6])

Let A be a C-algebra and let ½ 2 P(A). Then*

(i) $L_{½}$ = *fx 2 A* : ½($x^{¤}x$) = 0*g is a maximal norm closed left ideal of A;*

(ii) $R_{½}$ = *fx 2 A* : ½($xx^{¤}$) = 0*g is a maximal norm closed right ideal of A;*

(iii) $L_{½}$ + $R_{½}$ = *ker½;*

(iv) ½($L_{½}$) = ½($R_{½}$) = 0*;*

(v) $L_{½}$ *and* $R_{½}$ *are precisely the maximal norm closed left and right ideals of A, respectively, as ½ ranges over P(A).*

4.2.8 Let A be a C*-algebra with norm closed inner ideal I. From the classification provided above, in (4.2.7), it is simple to observe that I is equal to A if and only if it is not annihilated by any pure state of A. Indeed, if I is not equal to A then, as the intersection of closed left and right ideals, I is contained in $L_{½}$ or $R_{½}$, aAnprdeoisaiehydthe maximainedso

Proposition 4.2.10

Let A be a JB-triple and let J be a norm closed ideal of A. Suppose that $J^{**}_{at} = A^{**}_{at}$. Then $J = A$.*

Proof

By (4.2.5(a)(b)), $\rho(J) \neq 0$ for all $\rho \in \partial_e(A^*_1)$. Suppose that J is not equal to A, so that there exists some non-zero $\rho \in \partial_e((A/J)^*_1)$. Then, via the bijection (2.10.9(b)(iii)) between $\partial_e((A/J)^*_1)$ and the extreme points of A^*_1 vanishing on J, we obtain, in contradiction, $\tilde{\rho} \in \partial_e(A^*_1)$ such that $\tilde{\rho}(J) = 0$. $\square$

4.2.11 Let A be a JB*-triple with norm closed inner ideal I. Recall that $T(I)$ denotes the norm closed ideal of A generated by I. Since $I^{**}_{at} \subset T(I)^{**}_{at} \subset A^{**}_{at}$, we have the following corollary of (4.2.10).

Corollary 4.2.12

Let A be a JB-triple and let I be a norm closed inner ideal of A such that $I^{**}_{at} = A^{**}_{at}$. Then $T(I) = A$. That is, A is generated as a norm closed ideal by I.*

4.2.13 Let I be a norm closed inner ideal of a JB*-triple A and suppose that A^{**} and I^{**} have equal atomic part. The role of the next proposition is, essentially, to illustrate how this supposition translates to the quotient of A by any norm closed ideal, J say, and also to the intersection, $I \cap J$. Importantly, we also show that this constraint is preserved by triple homomorphisms π, that is, if $I^{**}_{at} = A^{**}_{at}$ then $\pi(I)^{**}_{at} = \pi(A)^{**}_{at}$.

Proposition 4.2.14

4.2.15 Recall that if A is a JB*-triple, then A^{**}_{at} is an ℓ_∞ direct sum of Cartan factors, $A^{**}_{at} = \sum C_\alpha$, say. Furthermore, if I is an inner ideal of A with $I^{**}_{at} = A^{**}_{at}$, then, as in the proof of (4.2.5(a)),

$$\sum C_\alpha = A^{**}_{at} = I^{**}_{at} = \sum I^{**} \cap C_\alpha.$$

Therefore, using (3.5.2), this condition on the atomic part of I^{**} guarantees that A and I have the same Cartan factor representations in the following sense.

Lemma 4.2.16

Let C be a Cartan factor. Let A be a JB-triple with norm closed inner ideal I. Suppose that $I^{**}_{at} = A^{**}_{at}$. Then I has a Cartan factor representation onto C if and only if A has a Cartan factor representation onto C.*

4.3 The Finite Cartan Part

4.3.1 We shall say that a JBW*-triple M is of *finite Cartan factor type* if it is an ℓ_∞-sum of the form $\sum A_i \otimes C_i$, where the A_i are abelian von Neumann algebras and the C_i are finite dimensional Cartan factors. For any JBW*-triple M, we define the *finite Cartan part* of M, denoted by M_f, to be the largest weak* closed ideal of finite Cartan factor type contained in M.

4.3.2 Let I be a norm closed inner ideal of a JB*-triple A such that the atomic parts of I^{**} and A^{**} coincide. We shall now show the equality of the finite Cartan factor parts of I^{**} and A^{**}. In particular, in conjunction with (4.2.5), this means that if K and J are norm closed inner ideals of A such that $K \subseteq J$ and $\partial_e(K^*_1) = \partial_e(J^*_1)$, then we can conclude that $K^{**}_f = J^{**}_f$. Hence, to prove the Inner Stone-Weierstrass theorem, that is, that under these conditions $K = J$, the outstanding problem is then to show the coincidence of the remaining parts.

Proposition 4.3.3

Let A be a JB-triple. Let I be a norm closed inner ideal of A such that $I_{at}^{¤¤} = A_{at}^{¤¤}$. Then $I^{¤¤}$ and $A^{¤¤}$*

4.4 The Inner Stone-Weierstrass Theorem for Uni-

The spaces $H_{½} = fa\ 2\ A : ½(a^{¤}a) = ½(aa^{¤}) = 0g$, are the maximal proper

We shall now prove the Inner Stone-Weierstrass Theorem for C*-algebras.

Theorem 4.4.4

4.4.5 Let A be a JC*-algebra and let I be a norm closed inner ideal of A. In order to exploit the previous theorem for C*-algebras, (4.4.4), and ultimately prove the analogous result for JC*-algebras, we consider the embedding of A in its universal enveloping C*-algebra, $C^*(A)$. In the remainder of this chapter we employ the following notation. We shall use I^e to denote the norm closed inner ideal of $C^*(A)$ generated by I. We have the subsequent commutative diagram of canonical embeddings.

$$\begin{array}{ccccc} I & \hookrightarrow & A & \longrightarrow & A^{**} \\ \downarrow & & \downarrow & & \downarrow \\ I^e & \hookrightarrow & C^*(A) & \longrightarrow & C^*(A)^{**} = W^*(A^{**}) \end{array}$$

Here, the equality comes from [HaSt, 7.1.11].

In what follows we also re-use some earlier notation; if J is a weak* closed inner ideal in a JW*-algebra M, J^{ew} shall denote the weak* closed inner ideal of $W^*(M)$ generated by J.

Lemma 4.4.6

Let A be a JC-algebra with norm closed inner ideal I. Then $(I^{**})^{ew} = (I^e)^{**}$.*

Proof

By looking at the second duals, we have $I^{**} \subseteq (I^e)^{**}$, which is a weak* closed inner ideal of $C^*(A)^{**}$. Thus, by definition, $(I^{**})^{ew} \subseteq (I^e)^{**}$.

Conversely, I is contained in $(I^{**})^{ew} \cap C^*(A)$, a norm closed inner ideal of $C^*(A)$. So, again by definition, $I^e \subseteq (I^{**})^{ew} \cap C^*(A) \subseteq (I^{**})^{ew}$. The converse now follows by taking the weak* closure of I^e in $C^*(A)^{**}$. $\square$

Theorem 4.4.7 ([HaSt, §7],[Aj, §4])

Let M be a JW-algebra generating a W*-algebra W (in some $B(H)$).*

(a) *If M has no type I_2 part then M is type I, II or III if and only if W is type I, II or III, respectively.*

(b)

Next we consider the associated weak* closed inner ideals that are generated in the universal enveloping W*-algebra. We have

$$(I^{**})^{ew} z = (I^{**} z)^{ew} = (J^{**} z)^{ew} = (J^{**})^{ew} z;$$

so that, via (4.4.6), $(I^e)^{**} z = (J^e)^{**} z$.
Since $s(\rho) \leq z$, so that $\rho(z) = 1$, we have $\rho(az) = \rho(a)$ for all a in $C^*(A)^{**}$. Hence $\rho((I^e)^{**}) = \rho((I^e)^{**} z) = \rho((J^e)^{**} z) = \rho((J^e)^{**}) \neq 0$. So $\rho(I^e) \neq 0$, as required. $\square$

4.4.9 Let I be a weak* closed inner ideal of a universally reversible JW*-algebra M. Furthermore, suppose that neither I nor M has non-zero type I_1 part. There exist central projections $z_1, z_2 \in M$ with $z_1 + z_2 = 1$, and such that $M = Mz_1 \oplus Mz_2$, where Mz_1 and Mz_2 are, respectively, the type I and continuous parts of M. Then I has a similar decomposition into type I and continuous parts, given by Iz_1 and Iz_2, respectively.
The canonical involution α of $W^*(M)$, through restriction, gives rise to the canonical involution on $W^*(Mz_2) = W^*(M)z_2$. As M is universally reversible, $Mz_i = W^*(M)^{\alpha} z_i$ $(i = 1, 2)$ [HaSt, 7.3.3]. So, by (3.4.2), the continuous part of I is of the form "$eM\alpha(e)$".
The type I part of I can be further decomposed into the sum of its finite Cartan part and its complement, N, say. By construction, N is an ℓ^{∞}-sum of JW*-triples of the form $A \bar{\otimes} C$, where A is an abelian von Neumann algebra and C is a Cartan factor of infinite dimension. Since, except when C is rectangular, all such summands possess a unitary tripotent, we have $N = K \oplus R$, where K contains a unitary tripotent u, say, and where R is rectangular without type I_1 part. As a consequence, $K = uu^* K u^* u = eM\alpha(e)$, where e is the projection uu^* in $W^*(M)$. Through (3.4.25), R is also of the form "$eM\alpha(e)$".

Taking this discussion as our motivation, in the final preparatory step of this section we show that inner ideals of the form "$eM\phi(e)$" are determined by the weak* closed inner ideals they generate in $W^*(M)$. In fact, for those universally reversible JW*-algebra without symplectic type I part, this result is contained within the proof of (3.4.24).

Lemma 4.4.10

Let M be a universally reversible JW-algebra and let e be a projection in $W^*(M)$. Let $I = eM\phi(e)$. Then $I = I^{ew} \cap M$.*

Proof

Since I is contained in $eW^*(M)\phi(e)$ we have, by definition,

$$I^{ew} \subset eW^*(M)\phi(e).$$

However $I = eW^*(M)\phi(e) \cap M$. Indeed given $x \in eW^*(M)\phi(e) \cap M$ we have $x = ea\phi(e)$ for some a in $W^*(M)$, so that

$$x = \phi(x) = e\phi(a)\phi(e) = e\left(\frac{a + \phi(a)}{2}\right)\phi(e) \in I.$$

The conclusion is now clear. $\square$

4.4.11 We are now ready to prove the Inner Stone-Weierstrass Theorem for universally reversible JC*-algebras. Subsequently, we shall extend this theorem to all JB*-triples (5.4.9), included amongst which are all JC*-algebras and JB*-algebras. We make repeated use of the elementary fact that if U, V and W are subspaces of the same linear space, and are such that $U \subset V$, $U + W = V + W$ and $U \cap W = V \cap W$, then $U = V$.

Theorem 4.4.12

Let A be a universally reversible JC-algebra. Let I and J be norm closed inner ideals of A such that I ½ J. Suppose that $@_e(I^{¤}_1) = @_e(J^{¤}_1)$. Then $I = J$.*

Proof

By (4.2.5), $I^{¤¤}_{at} = J^{¤¤}_{at}$. We claim that it is enough to prove the theorem in the case when $A^{¤¤}$, $I^{¤¤}$ and $J^{¤¤}$

Now we consider N. Using type decomposition and Cartan factor structure, we can create a further decomposition into a continuous part, and a type I part (containing no part of finite Cartan type). As discussed in (4.4.9), each summand is of the form $eA^{**}\phi(e)$, for some projection e in $C^*(A)^{**}$, and thus N also has this form. Since a similar argument holds for K, we have projections $e, f \in C^*(A)^{**}$ such that $N = eA^{**}\phi(e)$ and $K = fA^{**}\phi(f)$. Now, through (2.10.21) and (4.4.6), together with (4.4.8), we observe that

$$M^{ew} \oplus N^{ew} = (I^e)^{**} = (J^e)^{**} = M^{ew} \oplus K^{ew}.$$

Thus $N^{ew} = K^{ew}$. It follows from (4.4.10) that

$$N = N^{ew} \cap A^{**} = K^{ew} \cap A^{**} = K,$$

and therefore $I^{**} = J^{**}$, so that $I = J$, as required. $\square$

One technicality, in conjunction with Cartan factor representation theory (3.5.6), is vital to our argument. Namely, we demonstrate that to deduce the Inner Stone-Weierstrass Theorem for a JB*-triple A, it is sufficient to show it holds for an ideal J of A and for the quotient, A/J. Through this technicality, our attention can be focused upon JC*-triples whose Cartan factor representations all have rank greater than two. Indeed, the Inner Stone-Weierstrass Theorem for JB*-triples can be constructed from the JC*-triple version, using this argument, since every JB*-triple A has an exceptional postliminal ideal J such that A/J is a JC*-triple (2.12.6). The postliminal case is covered by the Stone-Weierstrass Theorem for postliminal JB*-triples provided by Sheppard, [Shep3, 5.5]. Furthermore, through (3.5.6), we can isolate a non-zero ideal K of A such that K/J is postliminal, and all Cartan factor representations of J have rank greater than two.

In broad terms, to prove the Inner Stone-Weierstrass theorem for these particular JC*-triples, we make use of the version for universally reversible JC*-algebras (4.4.12). We use a composition series argument typical of [BuChZa2]. More precisely, in the most important stage of this chapter, for a JC*-triple whose Cartan factor representations all have rank greater than two, we identify a composition series with successive quotients that are isomorphic to inner ideals in a universally reversible JC*-algebra. Such an isomorphism may be of independent interest. The crucial step is the identification of a non-zero ideal, through [MoRod1], with which to commence the series.

5.2 Clifford and Z-Hermitian Type JB*-Triples

5.2.1 Our remarks in this section are overwhelmingly influenced by the pioneering interpretation by Moreno-Galindo and Rodriguez-Palacios, given in [MoRod1] and [MoRod2], of the algebraic work of Zel'manov. The deep achievement of [MoRod1] was the classification of prime JB*-triples (together with their real versions). This was derived by the introduction of new techniques into JB*-triple theory. It is some of these techniques that we exploit in order to penetrate structure theory. We show, see (5.3.9), that if a JC*-triple has only Cartan factor representations of rank greater than two, then it has a composition series in which successive quotients can be realised as inner ideals in a universally reversible JC*-algebra. This is an important step towards the proof of the general Inner Stone-Weierstrass Theorem. It might also be of some independent interest.

For clarity and in order to explain terms, we have extracted the following synopsis from [MoRod1], where full details can be found. We first remark that in place of the term "hermitian" used in [MoRod1][MoRod2] we have employed the invented term *Z-hermitian*. Our excuse for this is that throughout this thesis we use the term hermitian in a way that conflicts with its meaning in [MoRod1][MoRod2].

5.2.2 Given a set of indeterminates X, $A(X)$ denotes the free associative algebra generated by X and $ST(X)$ denotes the Jordan subtriple of $A(X)$ generated by X. The Jordan triple system $ST(X)$ is the free Jordan triple system generated by X. A certain distinguished ideal, the *Zel'manov ideal*, G, of $ST(X)$ leads to the following definitions.

Let A be a JC*-triple. Then A is said to be *Z-hermitian* if $G(A) \neq 0$, that is, if for some triple polynomial $p(x_1, \ldots, x_n) \in G$ there exist $a_1, \ldots, a_n \in A$ such that $p(a_1, \ldots, a_n) \neq 0$. If $G(A) = 0$ then A is said to be *Clifford type*. It is immediate from the definition that if A is a JC*-triple of Clifford type then so is every JC*-subtriple and quotient of A.

Theorem 5.2.3 ([MoRod1, 6.1])

Let C be a special Cartan factor. Then

(a) *C is of Clifford type if and only if C has rank at most two;*

(b) *C is of Z-hermitian type if and only if C has rank greater than two.*

Proof

Part(a) is [MoRod1, 6.1]. Part (b) is immediate from (a). $\square$

Proposition 5.2.4

Let A be a JC-triple. Then*

(a) *A is of Clifford type if and only if all Cartan factor representations of A have rank at most two;*

(b) *A is of Z-hermitian type if and only if A has a Cartan factor representation of rank greater than two.*

Proof

(a) Let $\pi : A \to C$ be a Cartan factor representation. If A is of Clifford type then C is of Clifford type through the argument of the second paragraph of the proof of [MoRod1, 7.3]. Thus the rank of C is at most two (5.2.3(a)).

On the other hand, suppose that C has rank at most two, and so, through (5.2.3), is of Clifford type. Let $p(x_1, \ldots, x_n)$ lie in the Zel'manov ideal and let $a_1, \ldots, a_n \in A$. As C is of Clifford type, we have

$$\pi(p(a_1, \ldots, a_n)) = p(\pi(a_1), \ldots, \pi(a_n)) = 0.$$

Therefore, if all Cartan factor representations of A have rank at most two, we must have $p(a_1, \ldots, a_n) = 0$, and so $G(A) = 0$, giving that A is of Clifford type.

(b) This is immediate from part (a). $\square$

5.3 A Composition Series Relating JC*-Triples to Universally Reversible JC*-Algebras

5.3.1 The purpose of this section is to show that every Z-hermitian JC*-triple, that is, a JC*-triple for which all Cartan factor representations have rank greater than two, has a composition series whose successive quotients can be realised as norm closed inner ideals in a universally reversible JC*-algebra. As in section (5.2), we again make essential use of the ideas and results of [MoRod1].

5.3.2 To begin with, we recall that a C*-algebra A has a *matricial decom-*

5.3.3

Theorem 5.3.4 ([MoRod1, 5.6])

Let A be a Z-hermitian JC-triple. Then there exists a C*-algebra B with a matricial decomposition $fB_{ij}g$ and an associated even swapping involution, ®, such that*

(a) *A contains a non-zero ideal $J \cong B^{®} \setminus B_{12}$;*

(b) *$B^{®} \setminus B_{12}$ generates B as a C*-algebra.*

We now come to the point of this discussion.

Lemma 5.3.5

Let $fA_{ij}g$ be a matricial decomposition of a C-algebra A. Then A_{12} is a norm closed inner ideal of A.*

Proof

By definition, A_{12} is norm closed and the multiplication rules give

$$\begin{aligned} A_{12}AA_{12} &= A_{12}(A_{11} + A_{12} + A_{21} + A_{22})A_{12} \\ &½\ A_{12}A_{21}A_{12}\ ½\ A_{12}: \end{aligned} \qquad 2$$

The following is slightly more general than we require.

Lemma 5.3.6

Let I be a norm closed inner ideal in a JC-triple A and let J be the norm closed ideal of A generated by I. Let every Cartan factor representation of*

Proof

Let $\pi : J \to C$ be a Cartan factor representation of J. Since I generates J, if π vanishes on I then it vanishes on J. Hence, the weak* closure D of $\pi(I)$ in C is a non-zero weak* closed inner ideal of C, and thus a Cartan factor. By hypothesis, D has rank greater than two. Therefore, C has rank greater than two. $\square$

Proposition 5.3.7

Let A be a JC-triple for which every Cartan factor representation has rank greater than two. Then A contains a non-zero norm closed ideal J that is isomorphic to a norm closed inner ideal in a universally reversible JC*-algebra.*

Proof

By (5.2.4) A is Z-hermitian. Therefore, by (5.3.4) and (5.3.5), there exists a C*-algebra B with an involution α and a norm closed inner ideal I, such that $B^{\alpha} \cap I$ generates B and is isomorphic to a norm closed ideal, J say, of A.

Now, since all Cartan factor representations of J

5.3.8 We can now prove the main result of this section. The implication is that most JC*-triple theory can be reduced to the study of norm closed inner ideals in universally reversible JC*-algebras.

Theorem 5.3.9

Let A be a JC-triple for which all Cartan factor representations of A have rank greater than two. Then A has a composition series of ideals, $(J_\lambda)_{0\leq\lambda\leq\alpha}$, such that $J_{\lambda+1}/J_\lambda$ is isomorphic to an inner ideal in a universally reversible JC*-algebra.*

Proof

By (5.3.7), A contains a non-zero norm closed ideal J_1 that is isomorphic to an inner ideal in some universally reversible JC*-algebra. Since all Cartan factor representations of A/J_1 must have rank greater than two also, it follows that there is a norm closed ideal J_2 of A, such that $J_1 \subset J_2$ and $J_1 \neq J_2$, with J_2/J_1 again of the required form. Proceeding by transfinite induction, the result follows. □

5.4 The Inner Stone-Weierstrass Theorem for JB*-Triples

5.4.1 We are now ready to establish the Inner Stone-Weierstrass Theorem for JB*-triples, that is, if A is a JB*-triple with norm closed inner ideals I and J, with I contained in J and such that the extreme dual ball points of I and J are identical, then I and J are equal. As already intimated in the introduction to this chapter, as a norm closed inner ideal is a JB*-triple in its own right it is sufficient to show that the aforementioned theorem holds when $J = A$. It only remains to bring together all the key techniques developed in the earlier sections of this chapter.

5.4.2 Before we begin this section in earnest, we establish a vital technicality that enables a useful reduction to be made. That is, we show that it will be enough to consider JC*-triples whose Cartan factor representations are all of rank greater than two. We first recall the conjecture we aim to a□rm.

Conjecture 5.4.3

Let A be a JC-triple and let I be a norm closed inner ideal of A. Suppose that $@_e(I_1^{¤}) = @_e(A_1^{¤})$. Then $I = A$.*

Proposition 5.4.4

Let A be a JB-triple and let J be a norm closed ideal of A. Then if the conjecture (5.4.3) holds for J and for $A{=}J$, it holds for A.*

Proof

Let I be a norm closed inner ideal of A such that $@_e(I_1^{¤}$

Hence, $(I \cap J^{-})/J^{-}_{0}$ is a norm closed inner ideal of J^{-}/J^{-}_{0} and, by (4.2.5) together with (4.2.14(a)(b)), we have

$$((I \cap J^{-})/J^{-}_{0})^{**}_{at} = (J^{-}/J^{-}_{0})^{**}_{at}.$$

Thus, using the relevant Inner Stone-Weierstrass Theorem, (4.4.12), in conjunction with (4.2.5), we see that $I \cap J^{-} = J^{-}$, so that $J^{-} \subset I$, a contradiction.
$\square$

5.4.7 Let A be a JB*-triple. By Friedman and Russo's result, (2.12.3), A has an exceptional ideal J such that A/J is a JC*-triple. Since J is postliminal, by (5.4.4) together with [Shep3, 5.5], it is enough to show that the conjecture holds for A/J, that is for a JC*-triple. Therefore the next result follows as a simple corollary of (5.4.6).

Theorem 5.4.8

Let A be a JB-triple and let I be a norm closed inner ideal of A. Suppose that $\partial_e(I^*_1) = \partial_e(A^*_1)$. Then $I = A$.*

Finally we reach our desired conclusion, the Inner Stone-Weierstrass Theorem for JB*-triples.

Theorem 5.4.9

Let A be a JB-triple. Let I and J be norm closed inner ideals of A with $I \subset J$. Suppose that $\partial_e(I^*_1) = \partial_e(J^*_1)$. Then $I = J$.*

Theorem 5.5.3

Let A be a JB-triple and let I and J be norm closed inner ideals of A. Then $I=J$ if and only if $@_e(I_1^{¤}) = @_e(J_1^{¤})$.*

Proof

Suppose that $@_e(I_1^{¤}) = @_e(J_1^{¤})$ so that $I_{at}^{¤¤} = J_{at}^{¤¤}$ by (4.2.5). Let $N($[illegible]$; J)$ denote the norm closed inner ideal in A generated by I and [illegible] and [illegible]enceforth let bar denote weak* closure. By atomic decomposition (2.1[illegible] (a[illegible]), we have

$I^{¤¤} = I_{at}^{¤¤} © (I_{at}^{¤¤})^{?}$ and $J^{¤¤} =$ *h2 (7.97Tf-0.9[illegible]7.29[illegible]D[(1)]TJ/F311.95Tf5.672.96TL*

Theorem 5.5.4

Let A be a JB-triple and let I and J be two norm closed inner ideals of A. Then I ½ J if and only if $@_e(I_1^{¤})$ ½ $@_e(J_1^{¤})$.*

Proof

We use the notation of the previous theorem, in particular bar will denote weak* closure.

Suppose that $@_e(I_1^{¤})$ ½ $@_e(J_1^{¤})$, so that $I_{at}^{¤¤}$ ½ $J_{at}^{¤¤}$ by (4.2.5). As noted in the fourth paragraph of (5.5.3), (and in the notation of (5.5.3)),

$$N(I;J)^{¤¤} \text{ ½ } \overline{N(I^{¤¤};J^{¤¤})};$$

and $J_{at}^{¤¤}$ ½ $N(I;J)_{at}^{¤¤}$ ½ $\overline{N(I_{at}^{¤¤};J_{at}^{¤¤})}$, which is contained in $J_{at}^{¤¤}$, since the atomic part of the bidual of I is contained in that of the bidual of J. Thus $N(I;J)$ and J are norm closed inner ideals of A whose second duals have identical atomic parts. It follows, from (5.4.9), together with (4.2.5), that I ½ $N(I;J) = J$, as claimed.

The converse is apparent. *2*

5.5.5 To conclude, we consider a reformulation of the previous theorem. Let A be a JB*-triple with norm closed inner ideal I. Let $\rho \in A^*$ and let $\rho|_I$ denote its restriction to I. By weak* continuity, we can identify ρ with its weak* continuous extension and, in this manner, identify the restriction of ρ to I^{**} with $\rho|_I$. Let $I^\# = \{\rho \in A^* : \|\rho|_I\| = \|\rho\|\}$, that is, the set of functionals of A with norm preserving restriction to I.

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