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Classification of instability modes in a model of aluminium reduction cells with a uniform magnetic field

by

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1. Introduction

Aluminium is produced by decomposing alumina dissolved in a molten cryolite (sodium aluminium fluoride) by an electric current I_0 of 350-500kA, which passes vertically down from the anode to the cathode (Fig. 1). As a result of this process, a two-layer fluid system is formed with molten aluminium at the bottom of the cell and a slightly lighter cryolite at the top. Aluminium is then siphoned from the cell periodically.

As cryolite is a very poor conductor, much of the energy supplied is wastefully released in the cryolite layer in the form of Joule heating. The desire to reduce the thickness of the cryolite layer, and thus to reduce th



FIG. 1. Simplified schematic diagram of the aluminium reduction cell with horizontal crosssection given by an arbitrary function (x, y) = 0

A particular, widely used, simplified model of the MHD interaction within the cells leading to the instability has been suggested in [1], [5]-[7] and later employed in [8]-[16]. The model is based on the assumptions that the external, vertical magnetic field is uniform, and that the cell is completely covered by the anode. Concerning the nature of the instability within the limits of this model, main conclusions made by previous authors may be summarised as follows.

First of all, the external magnetic fields are highly complex, spatially varying within the cell. It has been recognized, however, that the most dangerous component of the field is vertical. It destabilizes the cell even if it is uniform [6].

Secondly, for certain cell geometries there is a critical value of the dimensionless parameter

$$j_0 B_0 L^2 / H_1 H_2 (1) g$$

above which the disturbance of the interface starts growing in magnitude. In the above j_0 is the density of the supplied current, B_0 is the induction of the vertical component of the magnetic field, $L\,$ is a typical horizontal dimension of the cell, $H_{\rm 1}\,,$

This mechanism, however, has been questioned by Lukyanov, El and Molokov [12], who noted that the key coupling between gravity and the Lorentz force occurs at the sidewalls rather than inside the domain. It has been suggested that the instability in closed domains takes place as a result of multiple reflections and amplification of the waves at the sidewalls.

Further, an exact solution for a circle has been analysed in [12] for >> 1 for the

cathode are characterised by the electrical conductivities $_{1}$,

To summarize, for instability to occur it is essential not only that the boundary is present, but that it is electrically insulating [13], [14]. These observations are crucial for further understanding of the nature of the unstable modes, as well as for the explanation of the fact that some modes develop strong growth, while the other ones are either weakly unstable or stable. This will be shown for several geometries of the cell.

In what follows we analyse first the exact solutions for a half-plane and for a circle for >> 1, and then develop an asymptotic solution for an infinite channel drawing attention to the common features of the instability modes for these geometries. Finally we will discuss the results obtained both here and previously and will develop a unified view of the mechanisms of instability for each group of modes.

3. Half-plane

The most basic geometry with a sidewall present is a half-plane x < 0, where the twofluid layer is bounded by a sidewall at x = 0 (Fig. 2). For this geometry the boundary conditions (2) are:

$$-\frac{1}{x} = 0$$
, $-\frac{1}{x} = \frac{1}{y}$ at $x = 0$ (5a,b)

We are looking for a solution in the form of travelling waves

$$\hat{(x)} \exp(ik_y y \quad i \quad t), \qquad \hat{(x)} \exp(ik_y y \quad i \quad t), \qquad (6a,b)$$

where $k_v > 0$ is a real wavenumber, and is a complex frequency.

Substituting Eqs. (6a,b) into Eqs. (1) and boundary conditions (5a,b) yields:

$$\frac{d^2}{dx^2}$$
 k_x^2 , 0, $\frac{d^2}{dx^2}$ k_y^2 , (7a,b)

$$\frac{d}{dx}$$
 0, $\frac{d}{dx}$ $i k_y$ at $x = 0$ (8a,b)

where

$$k_x^2 = k_y^2.$$
 (9)

The solution is(6-25.6116 - 4.992 2 2 218 - 3.18898) TD(a)5 T(lize)5 T(ds(6-20 T suc)5 T(h)0152(a)5 T(way t 2 2

$$i \frac{k_x}{k_y} (C_1 \quad C_2) e^{k_y x} \quad \frac{1}{2} C_1 e^{ik_x x} \quad C_2 e^{-ik_x x} ,$$
 (12)

where $C_{1,2} = \frac{1}{2} + 1 = \frac{k_y}{k_x(2^2 - i)}$. One of the exponential terms in Eq. (11) is the incident wave on the sidewall with the angle of incidence $\frac{1}{2}$ $\frac{1}{2}$ given by the expression tan $\frac{k_y}{k_x}$, while the other one is a reflected wave.



FIG. 3. Reflection coefficient as a function of for several values of : 0.5 (1), 1 (2), 1.5 (3), 2 (4), 10 (5).



FIG. 4. Reflection coefficient as a function of for several values of the reflection angle: $= 5^{\circ}$ (1), 30° (2), 45° (3), 60° (4), 85° (5).

The reflection angle for which the maximum is reached varies in the range /4 /2

for 0. The reflection coefficient increases indefinitely as 0, $_{max}$ /2,

which implies that the short waves almost aligned with the sidewall are amplified most.

On the other hand,

$$_{max}$$
 1 2 ¹ ..., $_{max}$ /4 ... as , (16)

implying that the amplification of the waves reduces to zero as the magnetic field induction

increases.

For 1 and for tan O(1)

crests and troughs of the wave in the whole domain as shown schematically in Fig. 5b. Sufficiently far from the wall the channels for coincide with those for , and the solution tends to become that for the unbounded domain.

The exact expression for $\hat{(0)}$ is:

$$(0) \quad \frac{1}{2}_{i},$$
 (19)

i.e. (0) i 1 0 as . Thus, in the limit not only the normal, but also the

where $2^{3/2}\sqrt{1}\sqrt{16^2}$, $1^{2/2}\sqrt{1}\sqrt{16^2}$, $1^{2/2}\sqrt{1}\sqrt{16^2}$, $1^{2/2}\sqrt{1}\sqrt{16^2}$. Similar to Sec. 3.1, parameter $/k_y^2$ is defined with the wavelength in the direction of propagation scaled with 2.

Eqs. (9) and (22) yield two possible values for , namely:

$$/k_{y} = r i_{i}, \qquad (23)$$

The solution with positive and negative signs of r represents waves propagating in the +y and -y-directions, respectively. The growth rate of the disturbance is determined by the imaginary part of For any non-zero value of the disturbance propagating in the -y-

1

which formally places an upper limit on the value of \cdot . For = 0.03 this gives << 2000, i.e. the theory presented here is valid for all practical purposes.

Calculations by Kohno an 0 0 11.303^{ulok0.6s7dzormT}]

$$\frac{2}{t^2} = \frac{2}{x^2}$$
. (28)

Thus, the wall either 'radiates' or 'absorbs' a spatially decaying wave in the transverse direction. The source of 'radiation' is the Lorentz force on the right-hand side of the boundary condition (5b).

From Eq. (27) follows that the wave speed in the direction transverse to the wall equals to unity, i.e. to the phase speed of gravity waves. The longitudinal wave speed is equal to $k_y^{-1}(/2)^{1/2}$ ($/2)^{1/2}$



FIG. 6. (a) Functions (solid line), (broken line), and j_v (dotted line) at x = 0, t = 0 for z = 100, and $k_v = 1$ and (b) schematic diagram of the mechanism of instability

The source of the disturbance is the Lorentz force at the boundary, which results from the interaction between the *core* current with the background field. This current flows owing to the differences in the core potential, which is a *global* function. As the interface disturbance vanishes outside the boundary layer, the core current is purely horizontal. It drives the fluid globally from one part of the boundary to the other. Thus the core potential, shifted in phase to , synchronises the propagation of the interface disturbance generated locally at various parts of the boundary.

Most of the results in subsections 3.2 and 3.3 have been obtained in [13] and [14]. They have been reproduced here as part of the unified theory of the phenomenon.

4. Circular domain

Another exact solution for travelling waves, which demonstrates important features of the modes, now in a closed domain, may be obtained for a circular geometry. Assuming

$$(r) \exp(in \quad i \quad t), \quad (r) \exp(in \quad i \quad t), \quad (29a,b)$$

where

 k_r^2 ², (r) $CJ_n(k_r r)$, (r) k_r^2 (r) $n^{1}(1)r^n$, (30a-c)

J_n() is the Bessel function of the first kind [20], n45625496591697724(20)550.60(55252295520625638866598640718126

Now, for each value of *n* there is an infinite number of roots $k_r^{(n,l)}$, l = 1, 2, ... of the dispersion relation (31). Both real and imaginary parts of the roots are either positive, or equal in modulus but negative, as the dispersion relation is invariant with respect to the transformation k_r . In both cases the roots lead to the same solution. For compatibility with the half-plane problem we will be concerned with the latter roots, i.e. $\text{Re}(k_r) = -\text{Re}(\) < 0$, $\text{Im}(k_r) = -\text{Im}(\) < 0$ (Fig. 6).

As increases, for each value of *n* the roots with l = 1 behave differently from those with l = 2 (Fig. 7). For l = 1 both real and imaginary parts of $k_r^{(n,l)}$ grow in modulus. For l = 2, the real parts of the roots monotonically decrease in modulus, while the imaginary ones first grow, and then decrease. Thus there are two distinct groups of modes which are discussed below.

$$k_r \qquad \sqrt{i} \qquad \sqrt{\frac{1}{2}} (1 \quad i) , \qquad (34a)$$

$$_L \qquad \sqrt{i}$$

is a global function, and is linear in both x and y. The electric current in the core being independent of either x or y

4.2 Centre, or Sele modes

5. Infinite channel

Schematic diagram of an infinite channel is shown in Fig. 10. What is new here with respect to the geometries considered before is that there are two separated boundaries, each of

$$\hat{} \quad \frac{1}{2} C_3 e^{ik_x x} \quad C_4 e^{-ik_x x} \quad C_5 e^{k_y x} \quad C_6 e^{-k_y x}, \tag{41}$$

where $k_x^2 = k_y^2$, and C_3 - C_6 are constants.

The boundary conditions (38a-d) and the normalization condition (39) yield the spatial exponents k_x as the roots of the dispersion relation

16² 2 (cos cosh q 1) $(p^2 q^2)$ sinh $q \sin p p^2 (p^2 q^2)^2 \sinh q \sin p 0$

5.1 Wall modes

Consider roots 1 and 2 first for $\text{Re}(k_x) < 0$. In contrast to the flow in a circular domain, which is unstable for any non-zero value of , there is a threshold of instability for the waves in a channel. For $k_y = 1$ roots 1 and 2 remain real below $_{cr} = 1.6$. At $_{cr} = _{cr}$ an internal resonance occurs as a result of which the imaginary parts of k_x and appear and the flow becomes unstable.

As the two boundaries, located at x = 0 and x = 2 are not connected, two separate boundary layers are formed for >> 1. They are denoted by *L1* and *L2*, respectively (Fig. 10). Integrating Eqs. (45), and taking into account the normalization condition (39) yields

the general solution as follows:

 $\hat{}_1 C_7 \exp(i)$

^ exp 2 exp()

However, counter-propagating, unstable, anti-symmetric waves in (x-1) are not only admissible, but are directly relevant to the flow in a rectangular domain. Such a wave may be represented by a linear combination of the two solutions obtained in the above. This gives:

$$\frac{\exp(k_{y})}{2\cosh k_{y}} \frac{1}{1} \frac{\exp(k_{y})}{2\cosh k_{y}} \frac{2}{2}, \qquad (55)$$

6. Reflection mechanism

Here a unified view on the nature of stable or weakly unstable modes for various geometries is outlined. It allows to understand their properties in terms of wave reflection



FIG. 14. Schematic diagram of reflection of short waves in various geometries: half-plane (a), circle (b), channel (c), square (d), and rectangle (e). The wave is amplified at points A, and suppressed at points S.

6.2 Applications of the reflection theory

As has been mentioned at the end of Sec. 5, the origin of the centre modes for a circle and a channel is similar. The qualitative difference between the two geometries is that these modes are stable for a channel and weakly unstable for a circle. This difference may well be explained by the reflection mechanism.

First of all, it has been established in Sec. 3.1 that for real values of k_x , a wave reflected from a plane wall, as shown in Fig. 14a, would be amplified. If the wave approaches the wall in the reverse direction it would be suppressed.

For a circle, any wave reflected from the boundary in the counter-clockwise direction is amplified (Fig. 14b). Then a reflected wave travels towards the other point of the same boundary, and is amplified again. As a result of multiple reflections, the amplitude of the wave grows, and all the modes are unstable [12]. For a channel (Fig. 14c), a wave amplified at one of the walls, reflects from a different wall. At that wall the local angle of reflection changes to the opposite, and the wave is suppressed. Owing to symmetry, the total coefficient of reflection from the two walls is equal to unity, and the wave remains stable.

For a square (Fig. 14d) any wave propagating counter-clockwise is amplified as the 'rays' form a closed trajectory. This is why the equilibrium is unstable for any non-zero value of , similar to a circle.

Concerning a rectangle (Fig. 14e), only some of the modes would grow in amplitude with time, namely those able to form a closed trajectory. The one shown schematically in Fig. 14e would not be. This would ultimately depend on the aspect ratio of the rectangle. As the increasing MHD interaction changes the angle insulating boundary form the first group. These waves are most unstable with the growth rate being $O(^{1/2})$ as . The key element for the development of this type of instability is the Lorentz force which carries the fluid from the right side of the crests at the sidewall to the left side resulting in anti-clockwise propagation of the wave. As the fluid flows through the core, it is accelerated by an unopposed Lorentz force and this leads to the amplification of the wave and th of

References

[1]