The Legendre Transformation And Grid Generation In Two Dimensions

S. L. Wakelin

Numerical Analysis Report 12/91

Department of Mathematics
University of Reading
P. O. Box 220
Reading RG6 2AX

The Legendre transformation is used as the basis of a method to generate irregular triangular grids in two dimensions. The method also generates piecewise linear approximations to functions. Grids are created for several functions including the solution of ˆgu11]X–fIu11]X–[3temSjds

1 Introduction

In this report we present a method for generating an irregular two-dimensional grid and creating a piecewise linear approximation to a function on the grid by an application of the Legendre transformation. The method is also used to give an approximate representation of semi-geostrophic frontogenesis in which the numerical issues involved in the time evolution are discussed.

Section 2 deals with the methods of creating the grid and approximating functions using the Legendre transformation. In section 3 some background to the meteorological problem of semi-geostrophic frontogenesis is presented and an initial approximate solution is found. A method to integrate the solution in time is outlined in section 4. In section 5 problems associated with the boundaries of domains are discussed and a method to overcome them is proposed.

2 The Legendre Transformation in Two Dimensions

Given a function P of two variables x and z we seek a piecewise linear approximation $\hat{P}$ to P.

One method of creating such an approximation would be to form a regular triangular grid in the xz plane and then consider the plane above each triangular element which intersects the surface of P vertically above the three vertices of the triangle. The set of these triangular plates is a 'chordal' type approximation to the function P.

It may be possible to improve on this approximation by using an irregular triangular grid in the xz plane and/or seeking alternative values of the approximation above the vertices of each triangle instead of using the corresponding value of the function P. An irregular grid can be created using the Legendre transformation (cf. [1]).

In one dimension, an application of the Legendre transformation leads to the generation of an irregular distribution of nodes along an axis and a piecewise linear approximation to a function of one variable [2]. It was found that, on the interval between any two adjacent nodes x_{i-1} and x_i on the axis, the second derivative of the function to be approximated, u, was equidistributed such that

$$\int_{x_{i-1}}^{x_i} u''(x)\,dx = \text{ constant.} \tag{2.1}$$

A consequence of this was that the nodes of the approximation tended to cluster in regions where the second derivative of the function was largest. By analogy with the method in one dimension, it may be that an irregular two dimensional grid, generated using the Legendre transformation, exhibits a similar property and the nodes of the grid cluster in regions where the second derivatives of the function to be approximated are largest.

Returning to the two dimensional problem, consider the coordinates (m, θ, R), dual to the original coordinates (x, z, P). The Legendre transformation gives a relationship between these sets:

$$m = \frac{\partial P}{\partial x}, \quad \theta = \frac{\partial P}{\partial z} \tag{2.2}$$

$$x = \frac{\partial R}{\partial m}, \quad z = \frac{\partial R}{\partial \theta} \tag{2.3}$$

$$P + R = mx + \theta z. \tag{2.4}$$

So, given $P(x, z)$, m and θ can be found as functions of x and z from (2.2). Provided these functions are invertible it is possible to express x and z as functions of m and θ and, using (2.4), to find an expression for the dual function R in terms of m and θ.

One property of the Legendre transformation is that a part of a plane in one space transforms to a point in the dual space. Thus, if the surface in one space is to consist of triangular plates, the dual surface must consist of non-overlapping plates where the projections onto the $m\theta$ plane of no more than three plates meet at any point. In particular, a surface which consists of hexagonal plates is sufficient to ensure that the dual surface is made up of triangular plates. Adjacent hexagons transform to points in the dual space which may be thought of as two of the vertices of a triangle, and three hexagons whose projections onto the $m\theta$ plane have a common node transform to the three vertices of a triangle (see figure 1).

Generating a regular hexagonal grid in the $m\theta$ plane, forming a linear approximation to the function R, then performing a Legendre transformation back to the original space, gives a set of points in the xz plane which may be used as nodes for an irregular triangular grid, and a set of values for the approximation to P at each node. Then one piecewise linear approximation to P is the set of planes, one above each triangle, which pass through the approximate values of P corresponding to each node of the triangle.

A regular grid of hexagons is set up in the $m\theta$ plane. Over each hexagon the best fit plane approximation to the surface R may be found by minimising

$$\|R - mx - \theta z + P\|_2 \tag{2.5}$$

with respect to P, x and z. The best fit plane above each hexagon transforms to a point in the original space.

Consider hexagon i. To find the corresponding point (x_i, z_i, P_i) minimise

$$\|R - mx_i - \theta z_i + P_i\|_2 \tag{2.6}$$

over $_i$ $_i$ $_i$. This leads to the equations

$$
\begin{aligned}
_{D_i}(\quad\quad _i \quad\quad _i + \quad _i) \quad\quad &= 0 \\
_{D_i}(\quad\quad _i \quad\quad _i + \quad _i) \quad\quad &= 0 \\
_{D_i}(\quad\quad _i \quad\quad _i + \quad _i) \quad\quad &= 0
\end{aligned}
\qquad (2\ 7)
$$

where the integrals are over the th hexagon in the plane. This gives rise to the matrix equation

$$
{}^2 \quad\quad\quad {}^2 \quad\quad\quad \begin{matrix} _i \\ _i \\ _i \end{matrix} \quad =
\qquad (2\ 8)
$$

to be solved for $_i$ $_i$ and $_i$, where the integrals are again over the th hexagon. This is done for each hexagon.

The point in ( ) space, transformed from hexagonal plate in ( ) space, is joined by straight lines to all other points transformed from hexagonal plates neighbouring plate . In this way a piecewise linear approximation to the function is constructed from the triangular plates formed when three points are joined by three lines (see figure 1).

This process has been performed for several trial functions. Irregular triangular grids are generated in the plane and piecewise linear approximations to the functions are found. The grids generated for the trial functions

$$(\quad) = \ {}^2 + \ {}^5 \text{ and } \ (\quad) = \ {}^{-8x} + \ {}^{-8z}$$

4 4

1 1 2

1 2

1 2

2 1 1 1 1 2

where $_i(\ )$ is the gradient in the direction and $_i(\ )$ is the gradient in the direction of the surface ˆ

i

$$ {}_i \quad {}^T {}_i $$

i

$$ {}_i \quad {}_i \quad {}^2 $$
$$ {}_i $$

$$ {}_i \quad {}^T {}_i \quad {}^2 \quad {}^T $$
R, i

$$ {}_i \quad {}^T {}_i \quad {}^T \quad {}^T \quad {}^T $$
$$ {}_i $$

to be solved for and .

The equations could be solved by Newton's Method as follows. Let

$$_1(\quad) = 2 \sum_i ({}_i \quad {}^T {}_i) {}^T(\quad + \quad {}^T) + {}^T \tag{4.9}$$

$$_2(\quad) = \tag{4.10}$$

Then

$$\frac{\quad {}_1}{\quad} = 2 \sum_i ({}_i \quad {}^T {}_i)(\quad + \quad {}^T) \quad (\quad + \quad {}^T) \quad {}^T(\quad + \quad {}^T) \tag{4 11}$$

$$\frac{\quad {}_1}{\quad} = \quad {}^T \tag{4.12}$$

$$\frac{\quad {}_2}{\quad} = \tag{4.13}$$

$$\frac{\quad {}_2}{\quad} = 0 \tag{4.14}$$

The Jacobian matrix is given by

$$= \begin{matrix} \frac{\partial \mathbf{F}_1}{\partial \mathbf{R}} & \frac{\partial \mathbf{F}_1}{\partial} \\ \frac{\partial \mathbf{F}_2}{\partial \mathbf{R}} & \frac{\partial \mathbf{F}_2}{\partial} \end{matrix} \tag{4 15}$$

Let ^{k} ^{k} be previous guesses to the solution of (4.7). Improved values may be found by solving

$$^k \quad \begin{matrix} {}^{k+1} \\ {}^{k+1} \end{matrix} \quad \begin{matrix} {}^k \\ {}^k \end{matrix} = \begin{matrix} {}^k_1 \\ {}^k_2 \end{matrix} \tag{4 16}$$

for ${}^{k+1}$ ${}^{k+1}$, where the k superscript denotes quantities evaluated at ^{k} ^{k}. This can be simplified by using various properties of matrix (4.15), in particular the entries of matrix (4.14) are all zero and matrix (4.11) will have some zero rows and columns where the values of $\hat{}$ corresponding to several of the nodes in the momentum space do not contribute to the area function (4.4) of any triangle in the physical space.

$$\begin{matrix} {}^T_1 \\ {}^T_2 \end{matrix}$$

$$_1 \quad _2$$

$$_1$$

$$_2$$

$$\begin{matrix} {}^T_1 \\ {}^T_2 \end{matrix} \; {}^k \quad \begin{matrix} {}_1 \\ {}_0 \end{matrix} \; {}^{k+1} \quad \begin{matrix} {}_1 \\ {}_0 \end{matrix} \; {}^k \quad \begin{matrix} {}_1 \\ {}_0 \end{matrix} \; {}^k$$

$$_1 \quad _2 \qquad\qquad _2$$

where $\mathbf{R}_1$ is $(r-l\times 1)$, $\mathbf{R}_0$ is $(l\times 1)$, $\hat{\mathbf{F}}_1$ is $(r-l\times 1)$ and $\mathbf{F}_0$ is $(l\times 1)$, $(\mathbf{R}_1,\mathbf{R}_0)^T$ is a rearrangement of $\mathbf{R}$ and $(\hat{\mathbf{F}}_1,\mathbf{F}_0)^T$ is a rearrangement of $\mathbf{F}_1$. This leads to

$$A(\mathbf{R}_1^{k+1}-\mathbf{R}_1^k)+B_1^T(\boldsymbol{\lambda}^{k+1}-\boldsymbol{\lambda}^k) = -\hat{\mathbf{F}}_1^k \tag{4.19}$$
$$B_2^T(\boldsymbol{\lambda}^{k+1}-\boldsymbol{\lambda}^k) = -\mathbf{F}_0^k \tag{4.20}$$
$$B_1(\mathbf{R}_1^{k+1}-\mathbf{R}_1^k)+B_2(\mathbf{R}_0^{k+1}-\mathbf{R}_0^k) = -\mathbf{F}_2^k \tag{4.21}$$

to be solved for $\mathbf{R}_1^{k+1},\mathbf{R}_0^{k+1}$ and $\boldsymbol{\lambda}^{k+1}$. Manipulation of these equations gives explicit equations for $\mathbf{R}_1^{k+1},\mathbf{R}_0^{k+1}$ and $\boldsymbol{\lambda}^{k+1}$:

$$\mathbf{R}_0^{k+1}=\mathbf{R}_0^k+\left(B_2^T(B_1A^{-1}B_1^T)^{-1}B_2\right)^{-1}\left(-\mathbf{F}_0^k-B_2^T(B_1A^{-1}B_1^T)^{-1}(\mathbf{F}_2^k-B_1A^{-1}\hat{\mathbf{F}}_1^k)\right) \tag{4.22}$$

$$\boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k+(B_1A^{-1}B_1^T)^{-1}\left(\mathbf{F}_2^k+B_2(\mathbf{R}_0^{k+1}-\mathbf{R}_0^k)-B_1A^{-1}\hat{\mathbf{F}}_1^k\right) \tag{4.23}$$
$$\mathbf{R}_1^{k+1} = \mathbf{R}_1^k-A^{-1}\left(\hat{\mathbf{F}}_1^k+B_1^T(\boldsymbol{\lambda}^{k+1}-\boldsymbol{\lambda}^k)\right). \tag{4.24}$$

The iteration process is likely to be computationally expensive since matrix A, which must be inverted in the iteration, varies with the solution $\mathbf{R}$ and may need to be updated. The solutions at the previous time step could be used as the initial values for the iteration at the present time step, the iteration process being continued until the vectors $\mathbf{R}$ and $\boldsymbol{\lambda}$ change by less than some tolerance. This may converge to a solution for $\hat{R}$ after some iterations. The positions of the triangle nodes in the xz plane can be recovered from the gradient of this approximation over each hexagon, and the values of $\hat{P}$ at each node can be found from (2.8). In this way the approximate solution, $\hat{P}$, to the problem may be integrated forwards in time ([3],[4]), and by a projection of the solution on to the xz plane, the motions of air parcels may be studied.

5 Domain Boundaries

In this section we outline a problem with the domain boundaries under the Legendre transformation and suggest a method to rectify it.

In the meteorology problem, consider the domain Γ in the momentum space and the domain Ω in the physical space from which Γ is derived in (3.18). In section 3, the smallest rectangular region completely enclosing Γ is discretised into regular hexagons. Those hexagons which lie entirely within the boundary of Γ are used as the domain, Γ_0, for the transformation which creates the irregular triangular grid in the xz plane in physical space. The region covered by this triangular grid does not correspond exactly to the original domain Ω but the boundary of the region appears well conditioned. Thus the approximate solution is found on a domain which is different from the domain of the original problem.

In an attempt to make the region covered by the irregular triangular grid closer to Ω, those hexagons on the boundary of Γ_0 are extended outwards from Γ_0 so that the domain covered by the hexagons, some of them now irregularly shaped, is exactly Γ. However, under the transformation, this results in an irregular triangular grid in the plane with a badly conditioned boundary.

One possible, but untested, method which may create a triangular grid covering a region closer to Ω in size and shape is as follows. Discretise the physical domain Ω into regular triangles so that all internal nodes of the grid are vertices of six triangles. Use a piecewise linear approximation, $\hat{\ }$, of the function in physical space to create a grid of points in the momentum space by an appli-

Numerically the semi-geostrophic equations of motion can be solved easily in the dual momentum space. The problem of advancing the physical solution with time is that of solving a constrained minimisation, the minimisation being to approximately satisfy conservation of mass in the physical space subject to a constraint imposed by the necessary piecewise linearity of the function in the momentum space.

Problems exist with the boundaries - the approximate solution is derived on a domain which differs from that of the original problem. A possible remedy for this is proposed.

7 Acknowledgement

I acknowledge receipt of a S RC research studentship.

8 References

[1] M. J. Sewell. Maximum and Minimum Principles. A unified approach, with applications. Cambridge University Press (1987).

[2] M. J. Baines and S. L. Wakelin. quidistribution and the Legendre Transformation. Numerical Analysis Report 4/91, University of Reading.

[3] M. J. P. Cullen and R. J. Purser. Properties of the Lagrangian Semi-geostrophic quations. J. Atmos. Sci.,**46**, 2684-2697.

[4] S. Chynoweth and M. J. Baines. Legendre Transformation Solutions to Semi-geostrophic Frontogenesis. Proceedings of the 7^{th} International Conference on Finite lement Methods in Flow Problems (d. T. J. Chung, G. R. Kerr). UAH Press, Huntsville, Al.

Figure 1: Legendre transformation from 3 hexagonal plates to a triangular plate in the dual space.

Figure 2: Grid for $P(x,z) = x^2 + z^5$.

Figure 3: Grid for $P(x, z) = e^{-8x} + e^{-8z}$.

Figure 4: Set of seven nodes for $q_i(R)$.