Approximations to hannel Flows using Variational Principles

S. L. Wakelin

Numerical Analysis Report 10/92

Department of Mathematics
University of Reading
P. O. Box 220
Reading RG6 2AX

Abstract

Variational principles whose natural conditions are the steady state shallow water equations of motion are stated. The corresponding variational principles for quasi one-dimensional flows are derived. Discontinuous solutions of the shallow water equations are considered by formulating new variational principles, whose natural conditions include the relations between the flow variables immediately either side of a hydraulic jump.

Approximations to continuous and discontinuous flows in channels of varying breadths and constant equilibrium depths are calculated using finite dimensional versions of these variational principles. Approximations to continuous flows are calculated on fixed grids using both the quasi one-dimensional and the two-dimensional formulations. Methods of generating adaptive grids in one dimension using the variational principles are also studied and these allow grid dependent approximations to continuous and discontinuous quasi one-dimensional flows to be found.

1 Introduction

In a previous report [1] a number of variational principles for unsteady and steady continuous shallow water flows were derived. The continuous, differentiable functions which satisfy the variational principles also satisfy the shallow water equations of motion. Many principles are available because the governing physical laws can be expressed in different variables, and because each free principle can be used to formulate one or more constrained principles. The main purpose of this report is to use these variational principles to generate finite dimensional approximations to solutions of the shallow water equations for channel flows.

In [1] the variational principle devised by Luke [2] for fluid flow beneath a free surface was used as a starting point to form other principles for unsteady shallow water flows which were then modified for time-independent flows. In Section 2 of this report the shallow water equations of steady motion are stated and the corresponding variational principles for steady flows are given. By assuming that the flow is quasi one-dimensional, so that the flow variables are functions of one space coordinate only, corresponding variational principles associated with quasi one-dimensional shallow water flow can be derived from the two-dimensional versions given in [1]. This derivation is also described in Section 2.

In Section 3 consideration is given to variational principles for discontinuous shallow water flows. Under certain circumstances, when the outlet depth is specified to lie in a particular range of values, a hydraulic jump may occur in the flow. At such a point of discontinuity the differential equations which govern the flow are not applicable. The flow immediately in front of the jump is related to that immediately behind the jump by jump conditions which are statements of the governing physical laws at such discontinuities ([3]). Three jump conditions govern the flow at a discontinuity in shallow water. One condition is used to locate the position of the jump and the other two relate the flow variables either side of the jump. All three conditions may be incorporated in variational principles for discontinuous flows.

Approximations to continuous and discontinuous shallow water flows are sought using the derived functionals. The method used is to find functions in a prescribed finite dimensional space for which the functionals are stationary. In Section 4 finite dimensional expansions using finite element basis functions are substituted into the variational principles and used to derive approximations on a fixed grid to the flow variables in one and two dimensions.

Approximations to continuous flows on adaptive grids are also considered. The jump conditions, derived in Section 3 for discontinuous flows, can be applied to the approximate solution at the internal nodes of the grid to give an algorithm for generating grid dependent solutions. This process is given in Section 5. An alternative method, based on similar reasoning, but solving directly the requirement that the functionals are stationary with respect to variations in the positions of the grid points, is also given in Section 5. In the case of minimising a functional these two methods of generating grid dependent solutions should derive minimum values less than those derived for the solutions generated on fixed grids but still greater than the exact minimum. Similar remarks hold for the case of a maximum principle.

Attempts are also made to approximate discontinuous flows. Finite dimen-

Details are given in Stoker [3]. The conservation of momentum equation (2.4) is satisfied by $E =$ constant for continuous flows. Thus the energy E is considered to be a constant whose value is to be specified.

Let be a channel of slowly varying breadth $B(x)$ and consider a section of the channel occupying the interval $[x_e, x_o]$ of the x-axis. Then, under the above conditions, the flow can be assumed to be quasi one-dimensional in the x direction. The flow variables are functions of x alone. The $\boldsymbol{\nabla}.$ operator is replaced by $\frac{1}{B(x)}\frac{d}{dx}(B(x)\cdot)$ and $\boldsymbol{\nabla}$ by $\frac{d}{dx}$. The quasi one-dimensional counterparts of (2.3) and (2.5) are

$$
\begin{aligned}
(BQ)' &= 0 \qquad \text{conservation of mass,} && (2.6)\\
v &= \phi', && (2.7)
\end{aligned}
$$

where $'$ denotes $\frac{d}{dx}$. The one-dimensional version of the conservation of momentum equation, $E' = 0$, is satisfied exactly by the assumption that E is constant on $[x_e, x_o]$.

2.2 Variational Principles for Steady Shallow Water

In [1] the variational principle of Luke [2] for fluid with a free surface was used to derive four variational principles for unsteady shallow water flow by applying the shallow water approximation to the variables within the functional and performing changes of variables using (2.1) and (2.2). These principles were then reduced to corresponding principles for steady shallow water by assuming that all of the flow variables are independent of time. The resulting variational principles for steady shallow water are given by

$$
\begin{aligned}
\delta L_1(\mathbf{Q}, \mathbf{v}, \phi) &= \delta \left\{ \iint_D (p(\mathbf{v}, E) + \mathbf{Q}.(\mathbf{v} - \boldsymbol{\nabla}\phi))\, dx\, dy \right. \\
&\qquad \left. + \int_\Sigma C\phi \, d\Sigma \right\} = 0, && (2.8)\\
\delta L_2(\mathbf{Q}, d, \phi) &= \delta \left\{ \iint_D (r(\mathbf{Q}, d) + Ed + \phi \boldsymbol{\nabla}.\mathbf{Q})\, dx\, dy \right. \\
&\qquad \left. + \int_\Sigma \phi\,(C - \mathbf{Q}.\mathbf{n})\, d\Sigma \right\} = 0, && (2.9)\\
\delta L_3(\mathbf{Q}, \phi) &= \delta \left\{ \iint_D (P(\mathbf{Q}, E) + \phi \boldsymbol{\nabla}.\mathbf{Q})\, dx\, dy \right. \\
&\qquad \left. + \int_\Sigma \phi\,(C - \mathbf{Q}.\mathbf{n})\, d\Sigma \right\} = 0, && (2.10)\\
\delta L_4(\mathbf{Q}, d, \mathbf{v}, \phi) &= \delta \left\{ \iint_D (-R(\mathbf{v}, d) + \mathbf{Q}.\mathbf{v} + Ed + \phi \boldsymbol{\nabla}.\mathbf{Q})\, dx\, dy \right. \\
&\qquad \left. + \int_\Sigma \phi\,(C - \mathbf{Q}.\mathbf{n})\, d\Sigma \right\} = 0. && (2.11)
\end{aligned}
$$

The function p is defined by

$$
p(\mathbf{v}, E) = \frac{1}{2g}\left(E - \frac{1}{2}\mathbf{v}.\mathbf{v}\right)^2
$$

and has the values of pressure. The function can be described as the Lagrangian density (see [1]) and is defined by

$$(\quad) = \frac{1}{2}\,\frac{\quad}{\quad} \quad \frac{1}{2} \quad {}^{2}$$

The function can be called the Hamiltonian density and is defined by

$$(\quad) = \frac{1}{2} \quad {}^{2} + \frac{1}{2}$$

Finally, the function has the values of flow stress and ( ) is defined by eliminating and from the equations

$$= \frac{1}{2} \quad {}^{2} + \qquad = \qquad \text{and} \qquad = \quad + \frac{1}{2}$$

The function in (2.8)–(2.11) is a given function on the boundary Σ of and is regarded as a given constant in these principles.

The natural conditions of (2.8)–(2.11) include the equations of steady shallow

1 2 3 4

1 D Σ

o

e

$\underline{\quad e}$

s s s

s

$+$

$-$

quations (2.6) and (2.7) govern the motion in the channel away from the jump but at the discontinuity, where derivatives are not defined, the differential equations do not apply. The behaviour of the flow variables at a point of discontinuity is governed by the jump conditions (3.1), two of which can be derived as the natural conditions of variational principles for quasi one-dimensional flow as follows.

Consider first a general functional of the form

$$(\ _s \ ) = \int_{x_e}^{x_s} (\quad ') + \int_{x_s}^{x_o} (\quad ') + [\ (\quad)]_{x_e}^{x_o} \qquad (3\ 2)$$

where $= (\ _1(\ ) \ldots \ _n(\ ))^T$, $' = (\ _1'(\ ) \ldots \ _n'(\ ))^T$ and $_s \quad (\ _e \ _o)$ is a point at which any of the $_i$ may be discontinuous. The position of $_s$ is allowed to vary in the variational principle $= 0$ and gives rise to jump conditions. Using Taylor series $= 0$ gives

$$\int_{x_e}^{x_s} \sum_{i=1}^{n} \quad u_i \quad u_i'x \quad \sum_{j=1}^{n} \quad u_i'u_j \ _j' + u_i'u_j' \ _j'' \quad i$$

$$+ \int_{x_s}^{x_o} \sum_{i=1}^{n} \quad u_i \quad u_i'x \quad \sum_{j=1}^{n} \quad u_i'u_j \ _j' + u_i'u_j' \ _j'' \quad i$$

$$+ \sum_{i=1}^{n} \quad u_i' + u_i \quad i \ _{x_o} \quad \sum_{i=1}^{n} \quad u_i' + u_i \quad i \ _{x_e}$$

$$+ \sum_{i=1}^{n} \quad u_i' \quad i \ _{x_{s+}} \quad u_i' \quad i \ _{x_{s-}} + \ _s \quad _{x_{s+}} \quad _{x_{s-}} = 0 \qquad (3.3)$$

where $_{s+}$ is the side of the point $_s$ towards $_e$ and $_{s-}$ is the side of $_s$ towards $_o$. At the point $_s$, the total variation in $_i$ is given by $\hat{} \ _i \ _{x_s} = \ _i \ _{x_s} + \ _i'(\ _s) \ _s$ Substituting this into (3.3) gives the natural conditions of $= 0$ as follows.

$$_i : \quad u_i \ — \ u_i' = 0 \qquad = 1 \ldots \qquad [\ _e \ _s) \ (\ _s \ _o] \qquad (3.4)$$

$$i \ _{x_o} : \quad u_i' + u_i \ _{x_o} = 0 \qquad = 1 \ldots \qquad (3.5)$$

$$i \ _{x_e} : \quad u_i' + u_i \ _{x_e} = 0 \qquad = 1 \ldots \qquad (3.6)$$

$$\hat{} \ i \ _{x_s} : \quad u_i' \ _{x_s} = 0 \qquad = 1 \ldots \qquad (3.7)$$

$$_s : \quad \sum_{i=1}^{n} u_i' \ _i' \ _{x_s} = 0 \qquad (3.8)$$

The functionals in the one-dimensional variational principles (2.13)–(2.16) are all of the form (3.2) with , , and being identified with the $_i$ as appropriate. Thus the natural conditions of (2.13)–(2.16) are given by (3.4)–(3.8). The natural conditions caused by variations of the $_i$ in the domain, (3.4), and on the inlet outlet^fI[(fl(3.5(3(S[fffInoebIL3k[SflnoebILsam[SflnoebIL([SfloebIL3kos[SflnoebILbIk3)[kflqoebIL3k case in Section 2.2. In addition, there are jump conditions caused by the variation of $_s$, given by,

for the 'p' principle $[\quad]_{x_s} = 0$ and $[\ (\ (\quad) + \quad)]_{x_s} = 0$
for the 'r' principle $[\quad]_{x_s} = 0$ and $[\ (\ (\quad) + \quad)]_{x_s} = 0$
for the 'P' principle $[\quad]_{x_s} = 0$ and $[\quad (\quad)]_{x_s} = 0$ and
for the 'R' principle $[\quad]_{x_s} = 0$ and $[\ (\quad (\quad) + \quad + \quad)]_{x_s} = 0$

The first jump condition in each case is the conservation of mass flow across the discontinuity since, by hypothesis for quasi one-dimensional flow, the breadth is continuous. The same argument applies to the second jump condition in each case which states that there is no change in the value of the flow stress across the jump. That the conditions are the same can be seen by using the definitions of mass flow and energy, (2.1) and (2.2), to perform changes of variables.

The conditions $[\quad]_{x_s} = 0$ and $[\quad]_{x_s} = 0$ are the same as $(3\ 1)_1$ and $(3\ 1)_2$ and are consistent with the theory ([3]).

The constrained 'r' principle (2.19) will be considered for practical implementation. The functional for this case is

$$ {}_1(\quad) = \int_{x_e}^{x_s} (\ (\quad) + {}^{e}\ ) + \int_{x_s}^{x_o} (\ (\quad) + {}^{o}\ ) \qquad (3\ 9) $$

where $= \frac{CB_e}{B}$. The natural jump condition of $_1 = 0$ is

$$ (\ (\quad) + {}^{e}\ )_{x_{s+}} \quad (\ (\quad) + {}^{o}\ )_{x_{s-}} = 0 \qquad (3\ 10) $$

One method of generating approximations to discontinuous depth functions which will be given in Section 5 involves finding continuous solutions on $[\ _e \quad _s)$

s o s s

s e o

x_s x_s

=1

n

1

$\frac{\quad e}{}$

$\frac{}{}$ n h
1

$\frac{}{}$ $\frac{}{}$ n h h
1

The Jacobian, J, and the vector $\quad = (F_1, \ldots, F_n)^T$ are evaluated using the seven point Gaussian quadrature formula.

It is possible to deduce properties of the expected solutions using the fact that the Jacobian is the Hessian of L_1. From the definition of the function $r(\quad , d)$ in Section 2.2, the second derivative of $r(Q, d^h)$ is given by

$$r_{d^h d^h} = \frac{Q^2}{{d^h}^3} \quad g. \tag{4.8}$$

Substituting the definitions $Q = d^h v^h$ and ${v^h}^2 = 2(E \quad gd^h)$ into (4.8) gives

$$r_{d^h d^h} = \frac{2E}{d^h} \quad 3g. \tag{4.9}$$

The depth approximation d^h is termed critical at a point if $d^h = \frac{2E}{3g}$. This is in agreement with the definition of critical flow in the exact solutions and corresponds to a point where the velocity has the critical value $c_* = \quad \overline{gd}$. From (4.9) it can be seen that if the flow is subcritical, that is $d^h > \frac{2E}{3g}$, then $r_{d^h d^h} < 0$ and if the flow is supercritical, that is $d^h < \frac{2E}{3g}$, then $r_{d^h d^h} > 0$. From (4.4) the Jacobian has the form of a weighted mass matrix, where $Br_{d^h d^h}$ is the weight function. Thus if the approximated flow is subcritical in the domain throughout the iterations J is negative definite and the solution of (4.3) maximises L_1. Alternatively, if the iterated approximate flow is supercritical in the whole domain J is positive definite and the solution of (4.3) minimises L_1. If both subcritical and supercritical flow occur during the iterations then J may be indefinite.

Thus, given the energy, E, of the flow, the mass flow at channel inlet, C, and the breadth variation of the channel $B(x)$, finite element depth solutions can be generated for continuous flows which are either wholly subcritical or wholly supercritical in the domain except for a possible region of critical flow.

The algorithm is implemented on the equi-spaced grid given by

$$x_i = x_e + \frac{i \quad 1}{n \quad 1}(x_o \quad x_e) \qquad i = 1, \ldots, n, \tag{4.10}$$

where $x_e = 0$, $x_o = 10$ and $n = 21$. Two sets of basis functions are considered: the first, α_i^l $(i = 1, \ldots, n)$, leads to continuous piecewise linear approximations and the second, α_i^c $(i = 1, \ldots, n \quad 1)$, gives discontinuous piecewise constant approximations. The basis functions are defined by

$$\alpha_1^l(x) = \begin{matrix} \frac{x_2 - x}{x_2 - x_1} & x & [x_1, x_2] \\ 0 & x & [x_1, x_2] \end{matrix},$$

$$\alpha_i^l(x) = \begin{matrix} \frac{x - x_{i-1}}{x_i - x_{i-1}} & x & [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & x & [x_i, x_{i+1}] \\ 0 & x & [x_{i-1}, x_{i+1}] \end{matrix} \qquad = 2 \ldots \quad 1 \tag{4.11}$$

$$\,^l_n(\ ) = \begin{matrix} \frac{x - x_{n-1}}{x_n - x_{n-1}} & [\ _{n-1} \quad _n] \\ 0 & [\ _{n-1} \quad _n] \end{matrix}$$

subcritical solution, using $^{0}_{i} = 4$ ($= 1 \ldots$), is found after 5 iterations for critical flow and 3 iterations for non-critical flow.

Figure 2 shows the finite element depth approximations for critical and non-critical flows in a channel with breadth distribution $_1(\ ) (\ = 8)$. The top two solutions are approximations to subcritical flows and the other two approximate supercritical flows. Figure 2 shows a linear interpolation to the breadth function using the 21 grid points given by (4.10). The sides of the channel are almost parallel for much of its length where the breadthfhd4[3k))Xf]Xf]X][n53[fft[Lflcr[(flapprw5ereb]Xf]X

3

0

*

′

i

i

h n i i $i=1$

i

i

points x_i.

Let

$$L_2(\boldsymbol{\phi}) = \int_{x_1}^{x_n} Bp(\phi^{h\prime}, E)\,dx + CB_e\left(\phi^h(x_n) - \phi^h(x_1)\right) \tag{4.17}$$

where $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)^T$. The finite element approximation to the velocity potential is given by $\boldsymbol{\phi}$ such that $L_2(\boldsymbol{\phi})$ is stationary, that is,

$$F_i(\boldsymbol{\phi}) = \frac{\partial L_2}{\partial \phi_i} = \int_{x_1}^{x_n} Bp_{\phi^{h\prime}}\alpha_i{}'\,dx + CB_e\left(\alpha_i(x_n) - \alpha_i(x_1)\right) = 0 \; i = 1, \ldots, n. \tag{4.18}$$

The solution is found using Newton's method. The Jacobian, J, is given by

$$J(\boldsymbol{\phi}) = \{J_{ij}\} = \left\{\frac{\partial F_i}{\partial \phi_j}\right\} = \left\{\frac{\partial^2 L_2}{\partial \phi_i \partial \phi_j}\right\} = \left\{\int_{x_1}^{x_n} Bp_{\phi^{h\prime}\phi^{h\prime}}\alpha_i{}'\alpha_j{}'\,dx\right\}$$

and is the Hessian of L_2. As for the 'r' principle, the Jacobian can be used to give information about the approximations. From the definition of the function $p(\mathbf{v}, E)$, the second derivative of $p(\phi^{h\prime}, E)$ is given by

$$p_{\phi^{h\prime}\phi^{h\prime}} = \frac{1}{g}\left(\frac{3}{2}\left(\phi^{h\prime}\right)^2 - E\right).$$

Using the constraint $v = \phi'$, an approximation, v^h, to the velocity, v, is given by $v^h = \phi^{h\prime}$. The critical speed c_* is defined by $c_* = \sqrt{\frac{2E}{3}}$ so that subcritical solutions, with $v^h < c_*$, have $p_{\phi^{h\prime}\phi^{h\prime}} < 0$ and supercritical solutions, with $v^h > c_*$, have $p_{\phi^{h\prime}\phi^{h\prime}} > 0$. Thus, for subcritical flows J is negative definite and for supercritical flows J is positive definite.

Given an approximation, $\boldsymbol{\phi}^k$, to the solution $\boldsymbol{\phi}$ a hopefully more accurate approximation, $\boldsymbol{\phi}^{k+1}$, is given by Newton's method, that is,

$$\boldsymbol{\phi}^{k+1} = \boldsymbol{\phi}^k + \delta\boldsymbol{\phi}^k, \tag{4.19}$$

$$\text{where } J\left(\boldsymbol{\phi}^k\right)\delta\boldsymbol{\phi}^k = -\mathbf{F}\left(\boldsymbol{\phi}^k\right). \tag{4.20}$$

The process is repeated until

$$\max_i \left|\delta\phi_i^k\right| < \text{ tolerance.} \tag{4.21}$$

The Jacobian and the vector $\mathbf{F}$ are integrated exactly. The Jacobian is again tridiagonal and (4.20) is solved by Gaussian elimination and back substitution.

The initial approximation, $\boldsymbol{\phi}^0$, to the velocity potential is given by

$$\phi_i^0 = \frac{i-1}{n-1}v_0 \quad i = 1, \ldots, n \tag{4.22}$$

where v^0 is assigned a value which determines whether the solution being calculated will be the subcritical or the supercritical approximation. If $v^0 < c_*\frac{x_o - x_e}{n-1}$ the solution will be subcritical and if $v^0 > c_*\frac{x_o - x_e}{n-1}$ the solution will be supercritical.

The algorithm is implemented on the grid given by (4.10) where $_e = 0$,

$_o$

1 $^{-3}$

$_0$

$_0$

$_0$ $_0$

$_h$ $_{h'}$

$_e$

$_o$

$_h$ ^{n} $_{i=1}$ $_i$ $_i$ $_h$ ^{n} $_{i=1}$ $_i$ $_i$ $_h$ ^{n} $_{i=1}$ $_i$ $_i$

$_i$

3 $^{x_n}_{x_1}$ $_h$ $_h$ $_h$ $_{h'}$ $_h$ $_e$ $_h$ $_n$ $_h$ 1

1 $_n{}^T$ 1 $_n{}^T$ 1 $_n{}^T$

$\frac{3}{i}$ $\frac{3}{i}$ $\frac{3}{i}$

quations $(4.25)_3$ yield

$$-\int_{x_1}^{x_n} B\alpha_i{}' Q^h \, dx + CB_e \left(\alpha_i(x_n) - \alpha_i(x_1)\right) = 0 \quad i = 1, \ldots, n,$$

which may be rewritten as

$$\begin{aligned}
\sum_{j=1}^{2} Q_j \int_{x_1}^{x_2} B\alpha_1{}'\alpha_j \, dx &= -CB_e, \\
\sum_{j=i-1}^{i+1} Q_j \int_{x_{i-1}}^{x_{i+1}} B\alpha_i{}'\alpha_j \, dx &= 0 \qquad\qquad i = 2, \ldots, n-1, \\
\sum_{j=n-1}^{n} Q_j \int_{x_{n-1}}^{x_n} B\alpha_n{}'\alpha_j \, dx &= CB_e,
\end{aligned}$$

or as

$$A_Q \mathbf{Q} = \mathbf{C}_Q, \tag{4.26}$$

where A_Q is a constant $n \times n$ matrix and $\mathbf{C}_Q$ is a constant $n \times 1$ vector with only first and last entries non-zero. The matrix A_Q is of rank $n - 1$ and is singular but, using the boundary condition $Q_1 = C$, the solution of (4.26) is unique. A_Q is tridiagonal so $\mathbf{Q}$ is calculated using Gaussian elimination and back substitution.

quations $(4.25)_1$ yield

$$\int_{x_1}^{x_n} B\left(r_{d^h} + E\right) \alpha_i \, dx = 0 \quad i = 1, \ldots, n$$

which, once Q^h is known, can be solved for d^h in the same way as in Section 4.1.1.

quations $(4.25)_2$ give

$$\int_{x_1}^{x_n} B\left(r_{Q^h} - \phi^{h\prime}\right) \alpha_i \, dx = 0 \quad i = 1, \ldots, n,$$

which may be rewritten as

$$\begin{aligned}
\sum_{j=1}^{2} \phi_j \int_{x_1}^{x_2} B\alpha_1\alpha_j{}' \, dx &= \int_{x_1}^{x_2} Br_{Q^h}\alpha_1 \, dx, \\
\sum_{j=i-1}^{i+1} \phi_j \int_{x_{i-1}}^{x_{i+1}} B\alpha_i\alpha_j{}' \, dx &= \int_{x_{i-1}}^{x_{i+1}} Br_{Q^h}\alpha_i \, dx \quad i = 2, \ldots, n-1, \\
\sum_{j=n-1}^{n} \phi_j \int_{x_{n-1}}^{x_n} B\alpha_n\alpha_j{}' \, dx &= \int_{x_{n-1}}^{x_n} Br_{Q^h}\alpha_n \, dx,
\end{aligned}$$

or as

$$A_\phi \boldsymbol{\phi} = \mathbf{C}_\phi,$$

where A_ϕ is an $n \times n$ matrix and $\mathbf{C}_\phi$ is an $n \times 1$ vector. Once ϕ^h and d^h are known $\boldsymbol{\phi}$ can be calculated directly. The matrix A_ϕ is of rank $n - 1$ and singular but ϕ is a potential function and the important quantity is its gradient so one of the values, say ϕ_1, can be specified arbitrarily. This procedure is equivalent to setting the arbitrary constant in ϕ by assigning its value at the boundary.

Results for critical flow in a channel with breadth $_1(\ )\ (\ \ = 4)$, given by (4.13), are shown in Figure 5. The energy is taken to be 50. The piecewise linear approximation to mass flow is given in Figure 5 . The piecewise linear approximations to the velocity potential and depth for a supercritical flow are given in Figures 5 and 5 respectively. Figure 5 shows an approximation to the velocity, the height of each dot representing the magnitude of the velocity over a particular element.

Results for the corresponding subcritical flow are given in Figure 6.

Thus the quasi one-dimensional variational principles of Section 2 can be used to generate finite element approximations to all the variables of shallow water motion. The methods developed so far in this section are now extended to give an algorithm for generating such approximations in two-dimensions.

The methods of Section 4.1 are extended and used on the constrained 'p' principle (2.12) to generate two-dimensional approximations to the velocity potential. The approach is essentially the same as the one-dimensional method.

The domain in two dimensions is approximated by a triangular grid. The finite

h

h $\quad p \quad i \quad i \quad i{=}1$

i

i

h

$4 \quad D^h \quad h \quad \Sigma_e \quad h \quad \Sigma_o \quad h$

$1 \quad p \quad T \quad e \quad o \quad h$

4

$i \quad \frac{4}{i} \quad D^h \quad \phi^h \quad i \quad \Sigma_e \quad i \quad \Sigma_o \quad i$

$ij \quad \frac{i}{j} \quad \frac{2 \quad 4}{i \quad j} \quad D^h \quad j \quad \phi^h \quad \phi^h \quad i$

k

$k+1$

$k+1$ k k

k k k

i $^{k}_{i}$

$i+(j \quad 1)n$ $\frac{i \quad 1}{n \quad 1}$ o e e

$i+(j \quad 1)n$ $\frac{1}{2} \frac{\quad 1}{\quad 1}$ $+(\quad 1)$

1

11

1 1 1

1

$\frac{x}{5}$ 2

4

$\frac{x}{5}$ 2

5

$\frac{3}{5}$ 2

$\frac{x}{3}$ 2

6 $\frac{3}{7}$ $\frac{x}{7}$ 2

0

0 i $\frac{i \quad e}{2 \quad 1}$ 0

0

0 $* \frac{x_o - x_e}{n-1}$

0 $* \frac{x_o - x_e}{n-1}$

e

$\frac{KB_e}{B_o}$ o

1 2 e 1 2

o 1 2 e o

4

-3

-1

0

0

one-dimensional approximation to ϕ, calculated in Section 4.1.2, as the initial approximation $\boldsymbol{\phi}^0$, that is,

$$\phi^0_{i+(j-1)n} = \hat{\phi}_i \quad i = 1, \ldots, n \ , \ j = 1, \ldots, m,$$

where $\hat{\boldsymbol{\phi}} = (\hat{\phi}_1, \ldots, \hat{\phi}_n)^T$ is the one-dimensional solution vector. Then, using $B_4(x)$ with $l = 5$, $n = 5$ and $m = 3$, the ratio $\frac{\max_i |\delta\phi_i^k|}{\max_i |\phi_i^k|}$ reached a minimum of 2.35×10^{-2} for 4 Newton iterations but increased as the iterations continued. The approximate solution at $k = 4$ is shown in Figure 9. This is the piecewise constant velocity approximation calculated from $\mathbf{v}^h = \boldsymbol{\nabla}\phi^h$ on each element, the length of the arrow in each element being directly proportional to the magnitude of $\mathbf{v}^h$. The failure of the method to converge completely in this case might be the result of the occurrence of a hydraulic jump in the supercritical flow for the shapes of channel considered. The approximation does exhibit certain properties of the exact solution in that the magnitude of the supercritical velocity decreases as the channel breadth decreases and the flow directions appear to be approximately consistent with the boundary conditions of zero flow across the channel sides and the line of symmetry $y = 0$.

Figures 10 and 11 show approximations $\mathbf{v}^h$ to the velocity for subcritical flows. The breadth variation in each case is $B_4(x)$ and several values of l are considered.

Figure 10 is the approximation for $l = 1$, $n = 9$ and $m = 7$. The increase in the magnitude of the velocity as the breadth decreases can be clearly seen and this agrees with the behaviour of exact subcritical solutions. Notice also that the speed is greater close to the x-axis than it is near the channel sides.

Figure 11 shows the consequences of increasing the lengths of the sections of the channel where the sides are parallel. In Figure 11a $l = 15$ and in Figure 11b $l = 5$. It can be seen that increasing l from 5 to 15 has very little affect on the approximation in the region between the lines $x = -5$ and $x = 15$. Also notice that the velocity in the domain $\{(x, y) : x \in [-15, -5], y \in [0, B(x)]\}$ is virtually constant and parallel to the channel sides.

This justifies the use of the boundary conditions $C(y) = -K$ on Σ_e and $C(y) = \frac{KB_e}{B_o}$ on Σ_o for $l >\sim 5$ which should ideally be applied at the ends of an infinitely long channel since they assume that the flow is uniform across the inlet and outlet cross sections.

Thus, the algorithm given in this section can be used to generate a two-dimensional piecewise constant approximation to the subcritical velocity in channels of various breadth distributions.

The methods of this section are all concerned with approximating continuous solutions of the shallow water equations of motion. In Section 5 approximations to discontinuous solutions are considered.

5 Solutions on Variable Grids

In order to be able to approximate discontinuous shallow water flows and to improve approximations to continuous flows, methods of generating grid dependent solutions are now considered.

First, the constrained 'r' principle (2.19) is used to generate a piecewise linear approximation to a discontinuous depth variation on a grid with just one adjustable node. This node is placed at the discontinuity by applying the jump conditions of Section 3. The method is modified slightly to generate approximations to continuous depth variations on variable grids. This idea is extended further to give a second algorithm for approximating discontinuous solutions. Finally, the constrained 'p' principle (2.18) is used to generate piecewise linear approximations to the velocity potential on a solution dependent grid generated directly from the variational principle.

5.1 The Constrained 'r' Principle

5.1.1 Grid with One Moving Node

Consider the constrained one-dimensional 'r' principle (2.19). Let the domain of integration, $[x_e, x_o]$, be split into $n-1$ adjacent intervals by the points x_i ($i = 1, \ldots, n$) given by (4.10). In order to approximate discontinuous flows one of these grid points must be chosen to be the initial approximation to the position of the discontinuity. This requires deducing which of the nodes is nearest to the actual position of the hydraulic jump. Let $N = n-1$ be the initial guess for the number of this node in the grid given by (4.10).

The basis of the method to approximate discontinuous flows is to generate approximate solutions in front of the jump and behind the jump and to couple the two approximations by means of a discontinuity at the position of the jump.

The energy E of the flow in front of the jump is given by the specified value E^e. Let the approximation to the depth in this region, $[x_1, x_N]$, be

$$d^e(x) = \sum_{i=1}^{N} d_i^e \alpha_i^e(x),$$

where

$$\alpha_1^e(x) = \begin{cases} \frac{x_2 - x}{x_2 - x_1} & x \in [x_1, x_2] \\ 0 & x \notin [x_1, x_2] \end{cases},$$

$$\alpha_i^e(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & x \in [x_i, x_{i+1}] \\ 0 & x \notin [x_{i-1}, x_{i+1}] \end{cases} \qquad i = 2, \ldots, N-1,$$

$$\phi^e_N(\ ) = \begin{cases} \frac{x-x_{N-1}}{x_N-x_{N-1}} & [\ _{N-1} \quad _N] \\ 0 & [\ _{N-1} \quad _N] \end{cases}$$

^{o}

N n

^{n}

^{o} ^o_i ^o_i

$i=N$

$$\phi^o_N = \begin{cases} \frac{x_{N+1}-x}{x_{N+1}-x_N} & N \quad N+1 \\ & N \quad N+1 \end{cases}$$

$$\phi^o_i = \begin{cases} \frac{x-x_{i-1}}{x_i-x_{i-1}} & i-1 \quad i \\ \frac{x_{i+1}-x}{x_{i+1}-x_i} & i \quad i+1 \\ & i-1 \quad i+1 \end{cases}$$

$$\phi^o_n = \begin{cases} \frac{x-x_{n-1}}{x_n-x_{n-1}} & n-1 \quad n \\ & n-1 \quad n \end{cases}$$

^{e} $^e_1$ ^e_N ^{T}

^{o} ^o_N ^o_n ^{T} N

^{e} ^o_N x_N x_1 ^{e} ^{e} ^{e} x_n x_N ^{o} ^{o} ^{o}

^{e} ^o_N

^{e}

^{e}

^{o}

^{o}

N

s

^{e} x_{s+} ^{o} x_{s-}

^{e} ^{e} ^{e} x_N ^{o} ^{o} ^{o} x_N

N

The equation

$$r\left(Q_s, d_N^e\right) + E^e d_N^e - r\left(Q_s, d_N^o\right) - E^o d_N^o = 0 \tag{5.3}$$

is solved to give the value, Q_s, of the mass flow which would occur at the jump if d_N^e and d_N^o were the actual depths of the flow before and after the jump. The 'r' principle (2.19), and therefore (5.1), is constrained to satisfy conservation of mass, that is,

$$Q(x)B(x) = CB_e \quad x \in [x_e, x_o]. \tag{5.4}$$

Since the breadth distribution is to be specified, (5.4) can be used to find the point x_s^N in the channel where the mass flow is of magnitude Q_s. Using the known breadth variation, $B(x)$, on the outlet section of the channel the value of x_s^N is found by bisection from

$$B\left(x_s^N\right) = \frac{CB_e}{Q_s}. \tag{5.5}$$

It is conjectured that the point x_s^N is closer to the actual jump position than the point x_N.

The algorithm for positioning a node at the jump is in two parts. Firstly, beginning with $N = n-1$ the corresponding value of x_s^{n-1} is found. Then, stepping backwards along the channel to the $n-2$ th node, the value of x_s^{n-2} is found. If $\left(x_{n-1} - x_s^{n-1}\right)\left(x_{n-2} - x_s^{n-2}\right) < 0$ then x_s lies between x_{n-2} and x_{n-1}. Otherwise the process is repeated until the node j is found such that $\left(x_j - x_s^j\right)\left(x_{j-1} - x_s^{j-1}\right) < 0$. Then, if $|x_j - x_s^j| < |x_{j-1} - x_s^{j-1}|$, the number, N, of the node to be moved to the jump position is j; otherwise $N = j-1$.

Once the number of the node to be moved to the jump position has been established (5.3) is used in an iteration process to position the node at the jump. The node x_N is moved to the position x_s^N which is calculated from (5.3) and (5.5). The finite element approximation to d is re-calculated on the modified grid and if (5.2) is still not satisfied (5.3) is solved for Q_s then (5.5) yields a new x_s^N. The node x_N is moved to x_s^N and the process is repeated until (5.2) is satisfied. The approximate solution has then been found and x_N is an approximation to the jump position.

The algorithm is applied to a grid with $x_e = 0$, $x_o = 10$ and $n = 21$. The breadth distributions considered here are

$$B_7(x) = 6 + 4\left(1 - \frac{x}{5}\right)^k \quad x \in [x_e, x_o], \qquad k = 2, 4.$$

The energy E is taken to be 50 and the mass flow at inlet C is assigned the value which causes the flow to become critical at the channel throat, that is, $C = 11.5$.

Under these conditions, for a tolerance on the Newton iteration of 10^{-3} and on the jump condition (5.2) of 10^{-3}, the method converges to a discontinuous approximation, with outlet depth specified to be 4.6, in 4 iterations on the position of the discontinuity once the node to be placed at the discontinuity has been found. These iterations require 11, 8, 6 and 6 Newton iterations. The initial values of the approximation at the nodes of the original, regular grid are $^{e}_{i} = 1$ ($= 1 \ldots$) and $^{o}_{i} = 4\ 6$ ($=$ $\ldots$). Once the number of the node to approximate the jump position has been deduced, subsequent approximations to the finite element solution use the approximation on the previous grid as the first guess in Newton's method to find the approximation on the new grid.

Figure 12 gives a result for $_{7}(\ )$ ($= 4$). Figure 12 is an approximation to discontinuous flow caused by specifying the outlet depth to be 4.68; the flow is supercritical before the jump and subcritical afterwards. The number of the node which has been moved to the jump position is 15. Figure 12 is the linear interpolation of the breadth function at the grid points.

The algorithm given in this section is a method for generating an approximation to a discontinuous flow in a channel, given the energy and mass flow at inlet, the breadth distribution and the depth at outlet. It uses a grid where all of the nodes except one are fixed. The algorithm causes the one movable node to be located at the position of the hydraulic jump.

In Sections 5.1.2 and 5.1.3 this method is extended and applied to a grid where all the internal nodes are allowed to vary.

In this section the constrained 'r' principle (2.19) is used to calculate a piecewise linear approximation to the depth on a solution dependent grid where all the internal grid points are allowed to move.

The domain of integration $[\ _{e}\ \ _{o}]$ is divided into 1 adjacent intervals by the points $_{i}$ ($= 1 \ldots$) given by (4.10). The method is to generate finite element approximations to the depth on each interval $[\ _{i}\ \ _{i+1}]$ ($= 1 \ldots$ 1) and use the jump condition at each internal node to reposition the node. Instead of just two finite element approximations coupled at a point, as in Section 5.1.1, there are now 1 solutions coupled at the 2 internal nodes.

Let $^{h}_{i}(\ )$ be the finite element approximation to in the th element, $[\ _{i}\ \ _{i+1}]$, given by

$$^{h}_{i}(\ ) = \ ^{L}_{i}\ ^{L}_{i}(\ ) + \ ^{R}_{i}\ ^{R}_{i}(\ ) \qquad = 1 \ldots \quad 1 \qquad (5\ 6)$$

where

$$\begin{array}{lll} ^{L}_{i}(\ ) & = & \begin{cases} \frac{x_{i+1}-x}{x_{i+1}-x_i} & [\ _{i}\ \ _{i+1}] \\ 0 & [\ _{i}\ \ _{i+1}] \end{cases} \qquad = 1 \ldots \quad 1 \\ ^{R}_{i}(\ ) & = & \begin{cases} \frac{x-x_i}{x_{i+1}-x_i} & [\ _{i}\ \ _{i+1}] \\ 0 & [\ _{i}\ \ _{i+1}] \end{cases} \qquad = 1 \ldots \quad 1 \end{array} \qquad (5\ 7)$$

e o i $i+1$

i $^{h}_{i}$

i $i+1$ $^{h}_{i}$

i $^{L}_{i}$ $^{R}_{i}$ T

i

e

o

e o

i

i

$^{L}_{i}$ 1 $^{L}_{i}$ x_i $^{R}_{i-1}$ 2 $^{R}_{i-1}$ x_i

e

1 2

o e

1 2

o

1 2

i

i $^{L}_{i}$ 1 $^{L}_{i}$ i $^{R}_{i-1}$ 2 $^{R}_{i-1}$

i i i

N

$n-1$ $^{n-1}_{s}$

$n-2$

$j-1$ $^{j-1}_{s}$ j $^{j}_{s}$ j $^{j}_{s}$

$j-1$ $^{j-1}_{s}$

With fixed the 1 finite dimensional approximations (5.6) are calculated on the new grid using the solution on the previous grid as the initial guess in Newton's method. If (5.11) is not satisfied for some in the range 1 . . . the grid points are repositioned using (5.12). The process is repeated until (5.11) is satisfied for all in the range 1 . . . . An approximation to a discontinuous shallow water flow has then been found.

The algorithm is applied on a domain where $_e = 0$, $_o = 10$ and the breadth distribution is given by

$$_9(\ ) = 6 + 4 \quad 1 \quad \frac{}{5}^{\ k} \qquad [\ _e \quad _o] \qquad = 2 \ 4$$

The number of grid points $= 21$. The energy is taken to be 50 and the mass flow at inlet is given the value which causes the flow to be critical at the channel throat, that is $= 11\ 5$.

Figure 14 gives solutions for the breadth distribution $_9(\ )$ ($= 2$). Figure 14 shows the depth approximation for an outlet depth of 4.6. The number of the node which is placed at the jump position is 15. Figure 14 is the piecewise linear interpolation to the breadth function using the 21 grid points. The position of the 15th node has moved to be at the jump position from its initial location. The other nodes have hardly moved, the curvature of the breadth function not being as large as, for example, that of the breadth function given by (5.10). Given a

$_e \quad _o \qquad _i \quad _{i+1}$

$_i$

^{n}

$_h \qquad _{i=1} \quad _i \ _i \quad _{i-1} \quad _i \quad _{i+1}$

$$
{}_{11}(\ ) \;=\; \begin{cases} 8 + 2\cos\frac{\pi x}{5} & [0\ \ 10] \\ 10 & [\ \ 5\ \ 0] \quad [10\ \ 15] \end{cases}
$$

$$
{}_{12}(\ ) \;=\; \begin{cases} 10 + 2\tanh 2(\qquad 8) + 2\tanh 2(2\qquad) & [0\ \ 10] \\ 10 & [\ \ 5\ \ 0] \quad [10\ \ 15] \end{cases}
$$

The energy is assigned the value 50 and the mass flow at inlet $= 11\ 5$. Let the criterion for convergence using (5.15) be that the value of (5.15) changes by less than 5%.

Consider a domain where the breadth is given by ${}_{11}$. Let the number of grid points $= 9$. Then the method converges to the subcritical approximation in 11 iterations on the nodal positions, to the supercritical approximation in 3 iterations and to a transitional approximation in 17 iterations.

The associated piecewise constant velocity approximations, calculated using ${}^{h} = {}^{h\prime}$, are shown in Figure 15. Figure 15 shows the supercritical approximation. The grid points have not moved from their original positions given by (4.10). The subcritical approximation is given in Figure 15 . The grid points have moved towards the midpoint of the channel, that is, towards the region where the curvature of the breadth is largest. Figure 15 shows an approximation to transitional flow where the flow is supercritical at inlet, becomes critical at the channel throat and then subcritical in the outlet section. Figure 15 shows the breadth variation.

Figure 16 gives corresponding results for a grid with 41 nodes. There is a slight node movement in the subcritical case (Figure 16) and none at all in the supercritical case (Figure 16).

I thank Dr. M. J. Baines and Dr. D. Porter for their help with this work. I acknowledge receipt of an S RC research studentship.

[4] R. Courant and D. Hilbert. Methods of Mathematical Physics Vol. 1. Interscience, 1953.

[5] G. H. Golub and C. F. Van Loan. Matrix Computations. The John Hopkins University Press, Baltimore, Maryland, U.S.A., 1989 (2nd d.).

[6] A. J. Wathen. Realistic igenvalue Bounds for the Galerkin Mass Matrix. I.M.A. J. Numer. Anal. **7** (1987) 449–457.

Figure 1: One dimensional basis functions a) piecewise linear and b) piecewise constant.

Figure 2: a) Depth approximations on a fixed grid and b) $B_1(x)$ ($k = 8$).

Figure 3: a) Depth approximations on a fixed grid and b) $B_3(x)$ $\nu = 7.5$, $\sigma = 1.5$.

Figure 4: a) $B_1(x)$ $(k = 2)$, b) supercritical velocity and c) velocity potential approximations on a fixed grid.

Figure 5: a)Mass flow, b) velocity potential , c) depth and d) velocity approximations on a fixed grid — supercritical case.

Figure 6: a)Mass flow, b) velocity potential , c) depth and d) velocity approximations on a fixed grid — subcritical case.

4Figure 7: xample of a two-dimensional grid for ( ) (= 2) wit
=7

Figure 8: Two-dimensional basis function.

Figure 9: Supercritical two-dimensional velocity approximation.

Figure 10: Subcritical two-dimensional velocity approximation.

Figure 11: Subcritical two-dimensional velocity approximations — $a)$ $l = 15$ and $b)$ $l = 5$.

Figure 12: $a)$ Depth approximation on grid with one moving node ($d_o = 4.68$) and $b)$ $B_7(x)$ ($k = 2$).

Figure 13: *a*) Subcritical depth approximation on an adaptive grid and *b*) $B_8(x)$.

Figure 14: a) Depth approximation on adaptive grid ($d_o = 4.6$), b) $B_9(x)$ ($k = 2$) and c) depth approximation ($d_o = 4.0$).

Figure 15: Adaptive grid with $n = 9$ a) supercritical velocity approximation, b) subcritical velocity approximation, c) transitional velocity approximation and d) $B_{11}(x)$.

Figure 16: Adaptive grid with $n = 41$ a) supercritical velocity approximation, b) subcritical velocity approximation, c) transitional velocity approximation and d) $B_{11}(x)$.