Direct Solution of Reservoir low Equations with Uncertain Parameters

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Abstract

This paper presents a direct method to determine the uncertainty in reservoir pressure, and other functions, using the time-dependent one phase 2- and 3-dimensional reservoir flow equations. The uncertainty in the solution is modelled as a probability distribution function. This is derived from probability distribution functions for input parameters such as permeability.

The method involves a perturbation expansion about a mean of the parameters. Coupled equations for second order approximations to the mean at each point and field covariance of the solution, are developed and solved numerically. This method involves only one (albeit complicated) solution of the equations, and contrasts with the more usual Monte-Carlo approach, where many such solutions are required.

The procedure is a development of earlier steady-state two-dimensional analyses and a transient mass-balance analysis using uncertain parameters.

These methods can be used to find the risked value of a field for a given development scenario.

1 Introduction

Difficulty in the mathematical and numerical modeling of physical systems, such as evaluation of the flow in underground oil reservoirs, may often arise when a precise knowledge of data is not available. Specifically, data that is crucial for describing the system, may only be known within certain limits of accuracy, or it may only be possible to specify certain statistical properties of the data. This may be due to inaccuracy in measuring equipment, or inaccessibility, and a high level of heterogeneity, in materials whose parameters are involved in the model equations,

It is the effects of these latter sorts of uncertainty on the solutions of analytic and numerical systems which form the basis of this research project. The usual approach to problems of this type is to use Monte-Carlo methods. How-

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$$\mathbf{1}\quad \mathbf{2}\qquad \frac{\mathbf{1}\quad 0\ \mathbf{1}\quad \mathbf{2}\quad 0\ \mathbf{2}}{k\ \mathbf{1}\ k\ \mathbf{2}}$$

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therefore treat the two-dimensional permeability field for a single realisation as a perturbation about some pre-defined mean value field,

$$k = k_0 + \alpha k_1. \tag{3}$$

We assume that $k_0 = \langle k \rangle$ is a deterministic mean, knowledge of which is available.

quation (1) can then be written,

$$\gamma \frac{\partial p}{\partial t} - \nabla((k_0 + \alpha k_1)\nabla(p)) = f_0(\mathbf{r}, t) + \alpha f_1(\mathbf{r}, t), \tag{4}$$

where p is the pressure solution for the specific realisation under consideration.

As in much work by Dagan, [2], and Dupuy and Schwydler, [3], we assume the pressure solution can be expressed in a series form

$$p = \sum_{m=0}^{N} \alpha^m p_m + R_{N+1}, \tag{5}$$

where R_{N+1} is the residue due to truncating the series for N^{th} order accuracy. Substituting equation (5) into (4), gives

$$\begin{aligned} &\gamma \frac{\partial}{\partial t}(\sum_{m=0}^{N} \alpha^m p_m + R_{N+1} \\ -\ &\nabla((k_0 + \alpha k_1)\nabla\left(\sum_{m=0}^{N} \alpha^m p_m + R_{N+1}\right) = f_0(\mathbf{r}, t) + \alpha f_1(\mathbf{r}, t). \end{aligned} \tag{6}$$

If we define p_0 to be the solution of the mean value problem, also known as the deterministic problem,

$$\gamma \frac{\partial p_0}{\partial t} - \nabla k_0 \nabla p_0 = f_0, \tag{7}$$

then, by equating successive powers of α, equation (6) can be split up into the $N+1$ set of hierarchical equations,

$$\gamma \frac{\partial p_0}{\partial t} - \nabla k_0 \nabla p_0 = f_0, \tag{8}$$

$$\gamma \frac{\partial p_1}{\partial t} - \nabla k_0 \nabla p_1 - \nabla k_1 \nabla p_0 = f_1, \tag{9}$$

$$\vdots$$

$$\gamma \frac{\partial p_m}{\partial t} - \nabla k_0 \nabla p_m - \nabla k_1 \nabla p_{m-1} = 0, \tag{10}$$

$$\vdots$$

$$\gamma \frac{\partial p_N}{\partial t} - \nabla k_0 \nabla p_N - \nabla k_1 \nabla p_{N-1} = 0, \tag{11}$$

$$\gamma \frac{\partial R_{N+1}}{\partial t} - \nabla (k_0 + k_1) \nabla \alpha R_{N+1} - \nabla k_1 \nabla p_N = 0. \tag{12}$$

This represents a set of coupled p.d.e.s for each admissible realisation. By truncating this series at the N^{th} term, we have imposed a level of accuracy on the possible solutions. In a statistical sense, we are not able to solve the $N+1^{th}$ equation (12), and so these equations are of N^{th} order accuracy. It may, of course, be possible to obtain bounds on the size of these residue terms over all admissible realisations. This would effectively give a measure of the accuracy of the hierarchical approximation.

2.2 Lognormal Distribution

If a Lognormal distribution function is assumed for the permeability, the expansion must be done about the geometric mean, [4]. This is equivalent to a linear expansion about the log of the permeability.

$$ln(k) = y = y_0 + \beta y_1,$$

where, $y_0 = \langle y \rangle$. So,

$$\begin{aligned} k &= e^{y_0} + \beta y_1 e^{y_0} + \frac{\beta^2 y_1^2}{2} e^{y_0} + \cdots \\ &= \kappa_g + \beta \kappa_1 + \beta^2 \kappa_2 + \cdots = \kappa_g + \sum_{m=1}^{\infty} \beta^m \kappa_m, \end{aligned}$$

where κ_g is the geometric mean.

Performing the same procedure, assuming the pressure has the form,

$$p = \sum_{m=0}^{N} \beta^m p_m + S_{N+1},$$

and substituting for pressure and permeability into equation (1), gives,

$$\gamma \frac{\partial}{\partial t}(\sum_{m=0}^{N} \beta^m p_m + S_{N+1}) - \nabla(\kappa_g + \sum_{m=1}^{\infty} \beta^m \kappa_m) \nabla (\sum_{m=0}^{N} \beta^m p_m + S_{N+1}) = f(\mathbf{r}, t). \tag{13}$$

Again, by equating powers of β we obtain the system of hierarchical equations,

$$\gamma \frac{\partial p_0}{\partial t} - \nabla \kappa_g \nabla p_0 = f_0 \tag{14}$$

$$\gamma \frac{\partial p_1}{\partial t} - \nabla \kappa_g \nabla p_1 - \nabla \kappa_1 \nabla p_0 = 0 \tag{15}$$

$$\frac{\quad}{\quad 2} \quad {}_0 \quad {}_2 \quad {}_1 \quad {}_1 \quad {}_2 \quad {}_0 = 0 \tag{16}$$

$$\vdots$$

$$\frac{\quad}{\quad m} \quad {}_g \quad {}_m \quad \sum_{i=1}^{m} \quad {}_i \quad {}_{m-i} = 0 \tag{17}$$

$$\vdots$$

$$\frac{\quad}{\quad N} \quad {}_g \quad {}_N \quad \sum_{i=1}^{N} \quad {}_i \quad {}_{N-i} = 0 \tag{18}$$

$$\frac{\quad}{\quad N+1} \quad {}_g \quad {}_{N+1} \quad \Big(\sum_{1}^{\infty} \quad {}^m \quad {}_m\Big) \quad {}_{N+1} \quad \sum_{i=1}^{N+1} \quad {}_i \quad {}_{(N+1-i)} = 0 \tag{19}$$

${}_1$ ${}_1$

$$\frac{\quad}{\quad 0} \quad {}_0 \quad {}_0 \quad {}_0$$

$$\frac{\quad}{\quad 1} \quad {}_0 \quad {}_1 \quad {}_1$$

$$\frac{\quad}{\quad 2} \quad {}_0 \quad {}_2 \quad {}_1 \quad {}_1$$

$$\frac{\quad}{\quad 3} \quad {}_0 \quad {}_3 \quad {}_1 \quad {}_3 \quad {}_1 \quad {}_2$$

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and,

$$\frac{{}^{n+1}_{2\ ij} \qquad {}^{n}_{2\ ij}}{\Delta} \qquad {}_h({}^{0}_{ij} \quad {}_h\ {}^{n}_{2\ ij}) \qquad {}_h({}^{1}_{ij} \quad {}_h\ {}^{n}_{1\ ij}) = 0 \qquad (33)$$

where the ( ) indices refer to spatial points (Δ Δ) in cartesian co-ordinates, and ${}^{n}_{z\ ij}$ refers to the numerical solution for ${}_z($ Δ), where is also in Cartesian co-ordinates.

Now let us denote a general value of the perturbation ${}_1$ at a discrete point (Δ Δ) by ${}^{1}_{i'j'}$, and consider the value at a second reference point, ($'$ $'$). Multiplying this into equation (32), and taking the mean values throughout the resultant, together with equations (31) and (33)), gives,

$$\frac{{}^{n+1}_{0\ ij} \qquad {}^{n}_{0\ ij}}{\Delta} \qquad {}_h({}^{0}_{ij} \quad {}_h\ {}^{n}_{0\ ij}) = {}^{n}_{0\ ij} \qquad (34)$$

$$\frac{{}^{1}_{i'j'}\ {}^{n+1}_{1\ ij} \qquad {}^{1}_{i'j'}\ {}^{n}_{1\ ij}}{\Delta}$$

$${}^{1}_{i'j'} \quad {}_h({}^{0}_{ij} \quad {}_h\ {}^{n}_{1\ ij}) \qquad {}^{1}_{i'j'} \quad {}_h({}^{1}_{ij} \quad {}_h\ {}^{n}_{0\ ij}) = {}^{1}_{i'j'}\ {}^{n}_{1\ ij} \qquad (35)$$

$$\frac{{}^{n+1}_{2\ ij} \qquad {}^{n}_{2\ ij}}{\Delta} \qquad {}_h({}^{0}_{ij} \quad {}_h\ {}^{n}_{2\ ij}) \qquad {}_h({}^{1}_{ij} \quad {}_h\ {}^{n}_{1\ ij}) = 0 \qquad (36)$$

This is now a complete set of coupled (numerical) p.d.e.s that can be solved. When these equations are being solved, simultaneously, the cross-correlation function is found, from equation (35), and then substituted into equation (36). In this form, it is a function of two (discretised) spatial points. The discretised autocorrelation function of the permeability field occurs in the ${}^{1}_{i'j'} \quad {}_h({}^{1}_{ij} \quad {}_h\ {}^{n}_{0\ ij})$ terms. These are basically just linear combinations of the autocorrelation parameters, with coefficients specifically dependent on the particular spatially-discretised scheme under consideration. The boundary conditions have been incorporated into the right hand side terms of the equations.

Performing the expansion for a lognormal distribution function, about the geometric mean, results in an extra term in the second order equation, as seen in equation (12). In discretised form, the set of coupled numerical equations becomes,

$$\frac{{}^{n+1}_{0\ ij} \qquad {}^{n}_{0\ ij}}{\Delta} \qquad {}_h({}^{0}_{ij} \quad {}_h\ {}^{n}_{0\ ij}) = {}^{n}_{0\ ij} \qquad (37)$$

$$\frac{{}^{1}_{i'j'}\ {}^{n+1}_{1\ ij} \qquad {}^{1}_{i'j'}\ {}^{n}_{1\ ij}}{\Delta}$$

$${}^{1}_{i'j'} \quad {}_h({}^{0}_{ij} \quad {}_h\ {}^{n}_{1\ ij}) \qquad {}^{1}_{i'j'} \quad {}_h({}^{1}_{ij} \quad {}_h\ {}^{n}_{0\ ij}) = {}^{1}_{i'j'}\ {}^{n}_{1\ ij} \qquad (38)$$

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${}^{0}_{\prime\,\prime} \quad {}_{\prime\,\prime} \quad {}^{1}_{1\ \prime\,\prime} \quad {}_{0kN\,\prime}$

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grid-points. quation (31) in this case becomes,

$$\frac{{}^{n+1}_{0\ ij} \quad {}^{n}_{0\ ij}}{\Delta}$$
$$+ \frac{({}^{0}_{i+1j} + {}^{0}_{ij})}{2\Delta\ ^2}\ {}_{0\ i+1j} + \frac{({}^{0}_{i-1j} + {}^{0}_{ij})}{2\Delta\ ^2}\ {}_{0\ i-1j}$$
$$+ \frac{({}^{0}_{ij+1} + {}^{0}_{ij})}{2\Delta\ ^2}\ {}_{0\ ij+1} + \frac{({}^{0}_{ij-1} + {}^{0}_{ij})}{2\Delta\ ^2}\ {}_{0\ ij-1}$$
$$\frac{({}^{0}_{i+1j} + {}^{0}_{i-1j} + 2\ {}^{0}_{ij})}{2\Delta\ ^2} + \frac{({}^{0}_{ij+1} + {}^{0}_{ij-1} + 2\ {}^{0}_{ij})}{2\Delta\ ^2}\ {}_{0\ ij} = {}_{0\ ij} \qquad (41)$$

The stability condition for this deterministic scheme is

$$\frac{4\Delta}{\ ^2} \quad 1 \qquad (42)$$

In this section we present some illustrative samples of the type of results that we have obtained using this method to solve the full statistical problem.

In each case we consider a single Fourier mode as the initial condition, with no flow conditions around the boundary, and zero forcing function. The region under investigation is square with unit length. All lengths and times are normalised for the purposes of this research.

Using a single Fourier mode as the initial condition means that in the case of a homogeneous mean value for the permeability, the solution to the p.d.e. under consideration, equation (1), may be expressed as the Fourier mode with an exponentially decaying amplitude,

$$(\quad) = {}^{-\pi^2 \frac{\langle k \rangle}{\gamma} t} \quad (\quad) \qquad (43)$$

It is fairly trivial to show by substitution that this is a solution to the model equation, satisfying the zero boundary conditions. We choose this test function as it is a straightforward solution whose deterministic behaviour is well-known.

The experiments performed have included using different values of , with both constant, and spatially-varying function forms. We also tried different sizes of variance, ${}^2_k$, and different correlation lengths, ${}_x$ and ${}_y$, for the P. A. F. Both the isotropic case, where ${}_x = {}_y$, and the anisotropic case, ${}_x = {}_y$ were considered.

Typical results of evolution can be seen in Figures 6.1 to 6.3. In this case, we have the case where the homogeneous mm)=l re('Doqm)'Doqm)dca)q=l'tHi'm)dcUerminiiI=H
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Figure 6.2(c)

Figure 6.2(d)

Figure 6.3(a)

Figure 6.3(b)

Figure 6.3(c)

Figure 6.3(d)

The deterministic solution, shown only at one time value, in Figure 6.1, behaves as expected, decaying exponentially, whilst retaining the basic shape given by = 0 0, and = 1 0. The maximum variance was seen to reach a maximum at around = 0 5, thereafter gradually decreasing, with the maximum variance concentrating in the corners whilst it decays. The second order correction to the mean, in Figures 6.3(a) to 6.3(d), begins by taking a similar shape to the deterministic solution, on a much smaller scale, of course. This value is much more subject to instabilities than the variance and deterministic approximations, and we see large increases for large time values.

Compared to experiments done assuming a higher mean value, we naturally see a correspondingly slower decay rate, for example, when = 0 1, the numerical decay rate is halved. The general shape assumed by the variance and second order approximations after one time unit are the same. The numerical value of the variance is, however, higher due to a greater relative spread in admissible realisations. There is a lower numerical value for $_2$ after the time interval. This may be due to the fact that $_2$ is related to the decay of the Fourier mode.

When we do experiments with a larger variance, ^{2} = 0 1, compared to Figures 6.2(a) to 6.3(d), as expected, we see both the variance and the cor-

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the specific point at which instabilities start to occur have yet to be done, but it has been observed that they can certainly be shown to occur when $+3$ can be shown to lie outside the stability range for our scheme.

[5] D. C. McKinney and D. P. Loucks, "Uncertainty Analysis Methods in Groundwater Modelling", Computational Methods in Subsurface Hydrology, 1990, pp 479-485.