A Conservative and Shape-Preserving Semi-Lagrangian method for the Solution of the Shallow Water Equations [1]

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Abstract

Semi-Lagrangian methods are now, perhaps, the most widely researched algorithms in connection with atmospheric flow simulation codes. In order to investigate their applicability to hydraulic problems, cubic Hermite polynomials are used as the interpolant technique. The main advantage of such an approach is the use of information from only two points. The derivatives are calculated and limited so as to produce a shape-preserving solution. The lack of conservation of semi-Lagrangian methods is however widely regarded as a serious disadvantage for hydraulic studies, where non-linear problems in which shocks may develop are often encountered. In this work we describe how to make the scheme conservative using an FCT approach. The method proposed does not guarantee an unconditional shock-capturing ability but is able to correctly reproduce the discontinuous flows common in open channel simulation without any shock-fitting algorithm. It is a cheap way to improve existing 1-D semi-Lagrangian codes and allows stable calculations beyond the usual CFL limits. A basic semi-Lagrangian method is presented that provides excellent results for a linear problem; the new techniques allow us to tackle non-linear cases whilst not unduly degrading the accuracy for the simpler problems. Two one dimensional hydraulic problems are used as test cases, water hammer and dam break. In the latter case, because of the non-linearity, special care is needed with the low order solution and we show the advantages of using Leveque's large time version of Roe's scheme for this purpose.

rithm connecting the regions of smooth flow[7].

In this paper we shall concern ourselves with the performance of shape-preserving Hermite cubic polynomials as the interpolant technique when implementing a semi-Lagrangian scheme to solve the 1-D shallow water equations. We will also consider the applicability and limitations of new ways of coping with the lack of the important property of conservation.

Below we shall briefly describe the semi-Lagrangian algorithm and the implementation of shape preserving solutions using Hermite polynomials. In Section 2 a technique to recover conservation will be introduced for the scalar case as well as for systems of equations. It will be pointed out that some difficulties can be met when applying it to a system of equations. In section 3 an extension will be proposed to overcome this difficulty.

To demonstrate its effectiveness, and to facilitate this study, two tests problems from the hydraulic literature have been selected and several numerical results are shown.

1.1 Test Problems

1. Dambreak flow

ven though the strategy to recover conservation in semi-Lagrangian schemes is not intended to produce a method able to cope with strong discontinuities, as in other shock-capturing methods, and our primary interests are in river and pipe flows, where discontinuities are generally weaker, the idealized dambreak problem was chosen because it is a classical example of non-linear flow with shocks to test conservation in numerical schemes and, at the same time, has an analytical solution.

This problem is generated by the one-dimensional shallow water equations given by

$$\begin{aligned} \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} &= 0 \\ \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A} + gI_1\right) &= 0 \end{aligned} \tag{1.1}$$

where A is the wetted cross sectional area, Q is the discharge and I_1 represents a hydrostatic pressure force term. For the ideal case of a flat, frictionless channel of unit width and rectangular cross section, we have $A = h$, h being the water depth, and $I_1 = \frac{1}{2}gh^2$, g being the acceleration due to gravity.

The initial conditions are

$$\begin{aligned} h(x,0) &= \begin{cases} h_L & \text{if } x \leq \frac{L}{2} \\ h_R & \text{if } x > \frac{L}{2} \end{cases} \\ Q(x,0) &= 0. \end{aligned}$$

Calculation times were used so as to avoid interaction with the extremities of the channel. The boundary conditions are then trivial.

2. The water hammer problem

The linearized water hammer problem, due to its popularity as an example in the related literature, was selected as a means to compare the performance of the monotone Hermite cubic interpolation against other proposed semi-Lagrangian schemes. It was also used to determine the extent of the effect caused by the recovery of conservation. Being a linear and simplified problem, it was suitable in order to focus attention on the adaptation of the algorithm to systems of equations. Following the dimensionless formulation of Sibetheros et al.[21], for instance, this second test problem deals with the solution of the linear system of equations

$$\begin{aligned} \frac{\partial H}{\partial t} + a\frac{\partial V}{\partial x} &= 0 \\ \frac{\partial V}{\partial t} + a\frac{\partial H}{\partial x} &= 0 \end{aligned}$$

with the initial conditions

$$H(x,0) = 0\ ,\ V(x,0) = 1$$

and the boundary conditions

$$H(0,t) = 0\ ,\ V(L,t) = 0$$

where

$$= \frac{\quad}{\quad}_0$$

The dimensionless variables are

$$= \frac{\quad}{_0}$$

$$= \frac{(\quad\ _0)}{_0}$$

Where is the velocity, is the specific head and 0 subscripts refer to the undisturbed values.

In this section we review the semi-Lagrangian solution to the scalar problem:

$$_t + (\quad) \quad = 0$$

which describes the advection of ( ). The invariance of a scalar quantity

$$(\quad) = (\ _0\ \ _0)$$

along a trajectory,

$$(\ ) = \ _0(\ _0) + \int_{t_0}^{t} (\quad) \tag{2 1}$$

is common to a wide variety of fluid dynamics topics. The aim is to obtain a good approximation of the function ( ) at all the $_i$ points of a fixed discrete grid, assuming that and are known everywhere in the grid at an earlier time $_0$.

In general, two distinct steps are involved. The first step determines the departure points $_0$ of the trajectories arriving at $_i$ from the past time through approximate solutions of (2.1). The second step is concerned with

$_0$

$_0$

and then limiting their values[5] in the manner

$$d_i = \text{sign}(d_i)\min(|d_i|\,,|3\Delta_{i-1}|\,,|\Delta_i|) \tag{2.3}$$

As explained by Rash and Williamson[18], (2.2) and (2.3) are more than a monotonicity constraint; they are also a form of convexity or positivity constraint in the sense that they control overshoots on the interval next to local extrema. Being actually nonmonotonic in such regions they prevent oscillations at the edge of flat regions. This can be useful in avoiding clipping of solutions and no new extrema are introduced.

Although there is an inevitable sacrifice of accuracy in the numerical result when monotonicity is sought, the above technique gives satisfactory results for many kinds of problems at a minimum computational cost.

As an illustration, Figs. 1 and 2 display some results from the solution of test case 2 with the described monotone Hermite cubic semi-Lagrangian method. They have been computed on two different grids of $N = 13$ and $N = 37$ points with $CFL = 1.5$. In all cases the continuous line is used in as a reference and it represents the solution using $CFL = 1$.

The upper parts in these pictures are the temporal variation of the head $H(L,t)$ at the downstream end. The corresponding (spatial) longitudinal head profiles for four dimensionless times (0.125, 1.125, 2.125 and 5.125) are shown in the lower parts of the same pictures. They have been arranged so that the thinner the line is, the greater the time it stands for. It can be seen that no oscillations are present in the solutions and that the accuracy increases with the number of points. These results compare very favourably with those published elsewhere[21].

Unfortunately, this method is not able to give satisfactory results when dealing with open channel flow problems in which discontinuities, such as bores or hydraulic jumps, may occur. This is a consequence of the nonlinearity of

lution U^H and a low order, shape-preserving, solution U^L,

$$U_i^M = \alpha_i U_i^H + (1 - \alpha_i) U_i^L \tag{3.1}$$

with $i = 1, N$ and

$$0 \leq \alpha_i \leq 1. \tag{3.2}$$

The coefficients $\{\alpha_i\}$ are to be chosen as large as possible whilst maintaining monotonicity. Obviously, a trivial solution exists when $\alpha_i = 0, i = 1, N$, corresponding to the already monotone U^L.

Denoting by U^n the solution obtained at the previous time level and by $\{U^n, i\}$ the set of solution values used to interpolate at the foot of the characteristic passing through x_i, the following inequalities provide adequate upper and lower bounds to the coefficients $\{\alpha_i\}$:

$$\min(\{U^n, i\}, U_i^L) \leq \alpha_i U_i^H + (1 - \alpha_i) U_i^L \leq \max(\{U^n, i\}, U_i^L). \tag{3.3}$$

quation (3.3) differs from the one given by Bermejo and Staniforth[2] only in that the value of the low order solution has been included in the bounds. When U^L is calculated using linear interpolation of the $\{U^n, i\}$ values for pure linear advection problems, the conditions reduce to those of Bermejo and Staniforth. For more general problems, including source terms or when solving non-linear systems, the presence of the low-order solution in (3.3) is necessary to allow new controlled extrema to be generated by the low order scheme. It is worth noting here that linear interpolation must be used with care when solving non-linear problems and it will be shown to be inadequate for non-linear systems with shocks.

We now proceed to discuss the recovery of conservation and, for the sake of clarity, we shall do it in the scalar case.

3.1 Scalar equations

The set of optimal $\{\alpha_i\}$ satisfying (3.3) and providing a monotone and accurate solution will be considered as upper bounds, $\{\alpha_i^{max}\}$, for another choice of $\{\alpha_i\}$ made to produce a monotone, accurate and conservative solution.

i $\quad {}^{max}_{i}$

M $\quad n$

n

i

i

i ${}^{H}_{i}$ ${}^{L}_{i}$ i ${}^{L}_{i}$ i $*$

i $\quad i$

i ${}^{H}_{i}$ ${}^{L}_{i}$ i

$'$

i i

i

${}^{max}_{i}$ i

i

i $\quad i$

The negative terms of the sum in (3.7), as well as those equal to zero, are supplied with the highest possible coefficient in order to reduce as much as possible the size of the total, i.e.,

$$_i \quad 0 \qquad _i = \quad {}^{max}_i$$
$$(\;) = 1$$

In order to calculate the coefficients for the rest of the terms, an estimate can be

$$^* \qquad _{iflag(i)=1} \quad _i \; _i$$

$$AV \qquad \frac{\qquad}{_{iflag(i)=0} \quad _i}$$

$$_i$$

$'$ $\qquad {}^{max}_i$

$$AV \quad {}^{max}_i \qquad _i \quad AV$$

$${}^{max}_i$$

$$AV \quad {}^{max}_i \qquad _i \quad {}^{max}_i$$

AV

$${}^{max}_i$$

(16)

i
i

(19)

(13 14)

$*$ 1 2

1 2

k k

$i+\frac{1}{2}$ $i+1$ i

k

k k k

$i+\frac{1}{2}$ $i+1$ i

k

k k k (19)

$\frac{1}{2}$ $\frac{1}{i+\frac{1}{2}}$

j j

1 $\frac{1}{j}$ 1

1 1 1 $\frac{1}{j}$ 1

1 $\frac{\Delta t}{\Delta x}$ 1 $i+\frac{1}{2}$ 1 1

(8)

(1)

$^{c}_{i}$ $^{(1)}_{i}$ $^{(1)L}_{i}$

$_{i}$ $_{i}$

$^{c}_{i}$ $^{(2)}_{i}$ $^{(2)L}_{i}$

$_{i}$ $_{i}$

0 1 L

0 $^{c}_{i}$ $^{(2)}_{i}$ 1 $^{c}_{i}$ $^{(2)}_{i}$ L $^{(2)L}_{i}$

$_{iflag(i)=0}$ $_{iflag(i)=1}$ $_{i}$

0

1 L

0

$^{(2)}_{i}$ L

0

$^{(2)}_{i}$

$^{c}_{ib}$

0

$^{c}_{is}$ $^{(1)}_{is}$ $^{c}_{ib}$ $^{(1)}_{ib}$ $^{(1)}$

must hold.

If it happens that

$$_{is}^{max} \quad {}_{is}^{(1)} \qquad {}^{(1)}$$

then ${}_{ib}^{c}$ can be completely removed from (4.4) so that

$${}_{ib}^{c} = 0 \qquad (\quad) = 1$$

and

$${}_{is}^{c} = \frac{{}^{(1)}}{{}_{is}^{(1)}} \qquad (\quad) = 0$$

Otherwise, ${}_{is}^{c}$ can be supplied with the largest possible value,

$${}_{is}^{c} = {}_{is}^{max} \qquad (\quad) = 1$$

and

$${}_{ib}^{c} = \frac{{}^{(1)} \quad {}_{is}^{max} \; {}_{is}^{(1)}}{{}_{ib}^{(1)}} \qquad (\quad) = 0$$

${}_{ib}^{c}$ ${}_{is}^{c}$

0 1

0

${}_{is}^{(2)}$ ${}_{ib}^{(2)}$

$${}_{is}^{c} \; {}_{is}^{(1)} \qquad {}_{ib}^{c} \; {}_{ib}^{(1)} \qquad {}^{(1)}$$

$${}_{is}^{c} \; {}_{is}^{(2)} \qquad {}_{ib}^{c} \; {}_{ib}^{(2)} \qquad {}^{(2)}$$

$${}^{(2)} \qquad L \quad \sum_{i \neq is, ib} {}_{i}^{c} \; {}_{i}^{(2)}$$

^{c}

$^{c}_{i}$

$_{i}$

$^{c}_{i}$

$^{c}_{i}$ $^{max}_{i}$

$^{c}_{i}$ $^{(1)}_{i}$

$_{i}$

$^{c}_{i}$ $^{(2)}_{i}$

$_{i}$

$_{i}$ $^{(1)L}_{i}$

^{2}

^{2} $_{i}$ $\overline{\quad ^{e}_{i} \quad ^{n}_{i}{}^{2}}$

	N = 12	N = 24	N = 48	N = 96
HP	0.629	0.384	0.332	0.269
MHP	0.795	0.489	0.387	0.304
IRC	0.684	0.535	0.415	0.322
IRCLP	0.718322			

2

2

2

(21)

max

Monotone Hermite cubic polynomials with derivatives calculated explicitly from the neighbouring points seems a very efficient method of interpolation in the context of semi-Lagrangian schemes.

A new recovery of conservation procedure has been proposed to render the scheme much more suited to hydraulic problems with shocks. It is based on an FCT approach and relays on the adequate choice of a set of coefficients combining a high and a low order method. Two hydraulic problems have been used as test cases, water hammer and dam break. In the latter case special care is needed with the low order solution and we have shown the advantages of using Leveque's large time version of Roe's scheme for this purpose.

In this paper we have been concerned with 1-D shallow water equations and so made use of the Riemann invariants. Most of the concepts presented here extend nevertheless to higher dimensions and to large systems. The semi-Lagrangian approach and the basic independent recovery algorithms work equally well in higher dimensions(6,13). They can clearly be applied to arbitrarily large systems. The in-phase conservation, in principle, could also be extended but there is a real danger of running out of degrees of freedom before conservation in all required variables is achieved.

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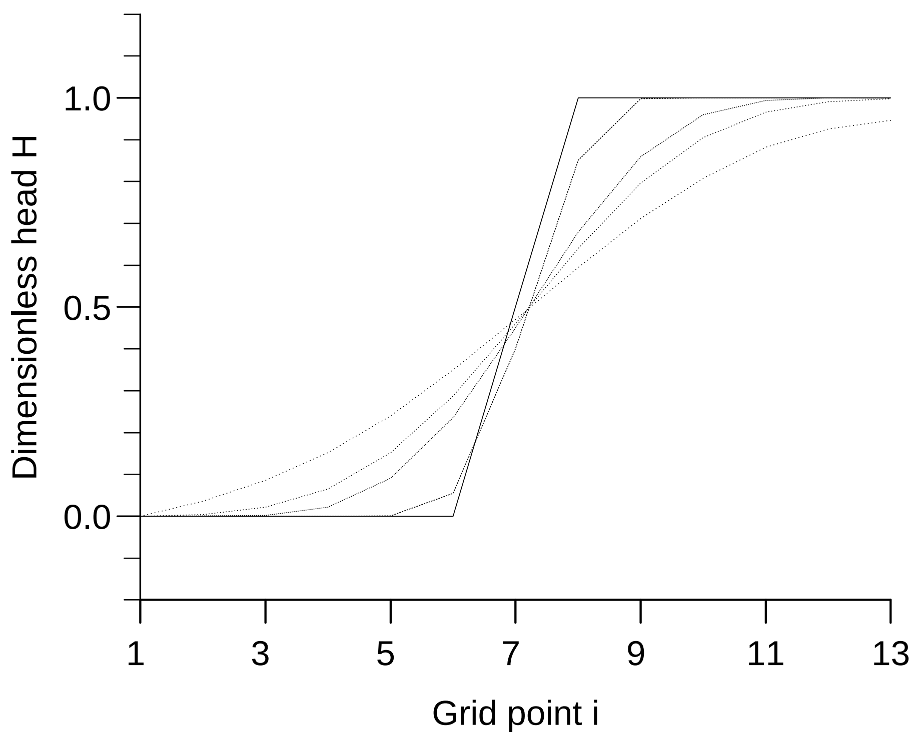

Dimensionless head H
1.0
0.5
0.0
1
3
5
7
9
11
13
Grid point i

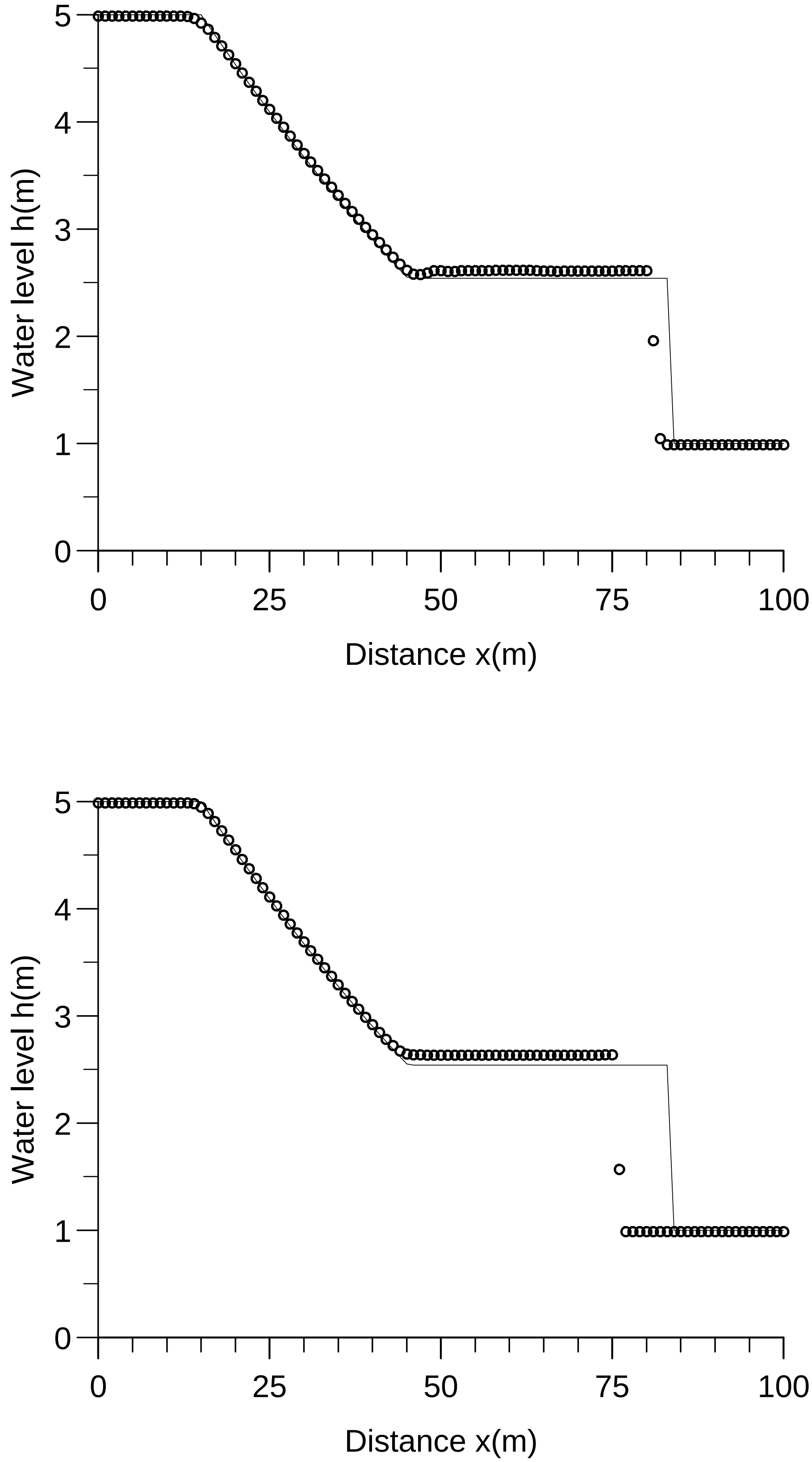
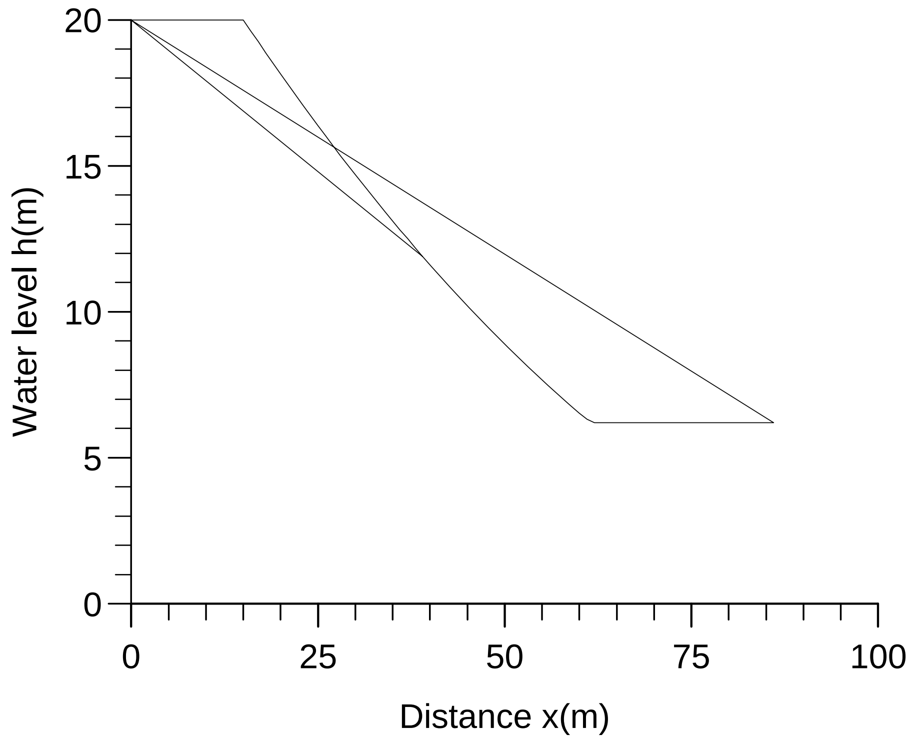


Figure 3: Monotone cubic interpolation. N=101. Dambreak problem for a height ratio 5:1. Upper: CFL=0.75. Lower: CFL=1.75


20
15
10
5
0
Water level h(m)
0
25
50
75
100
Distance x(m)

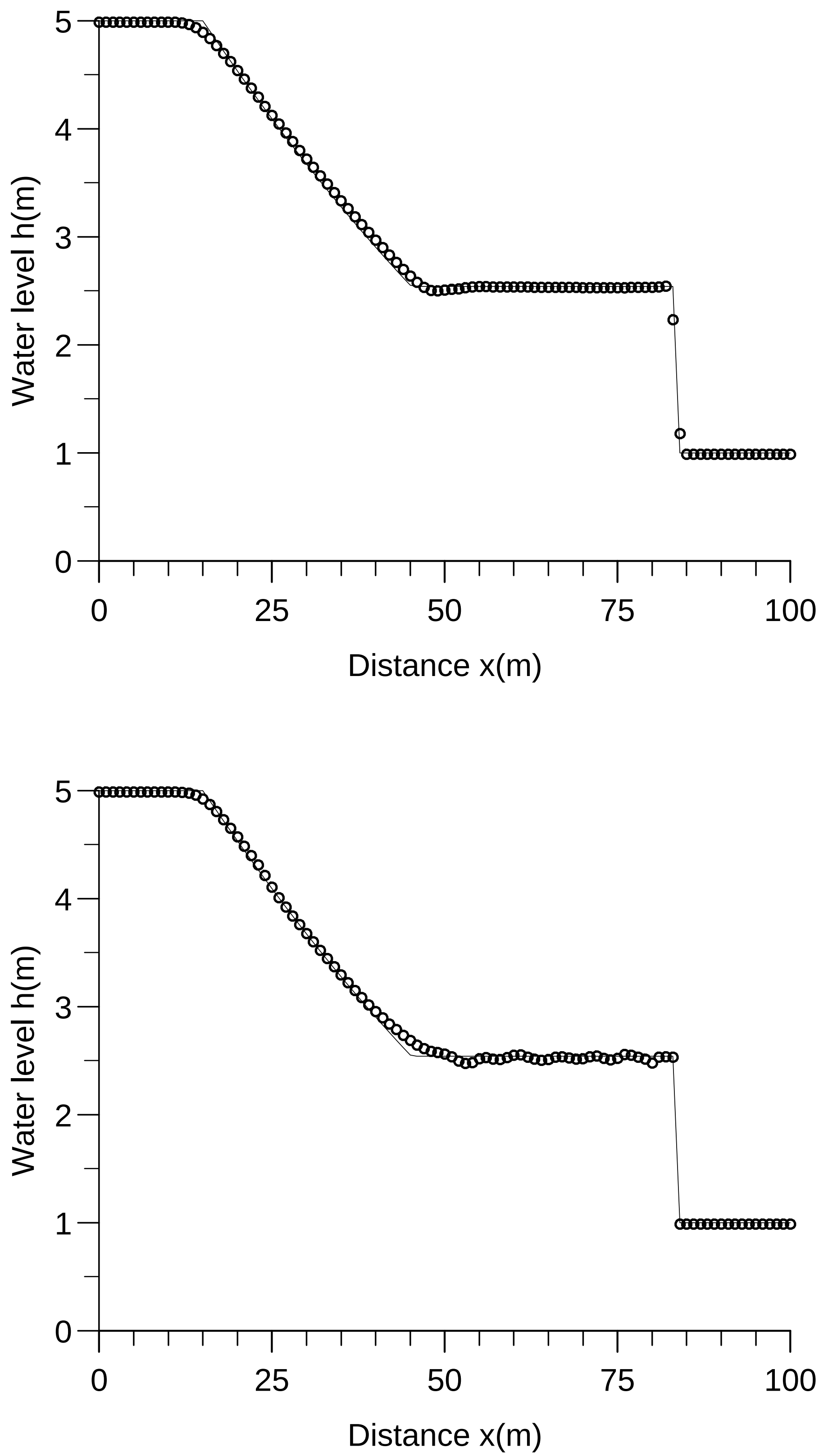
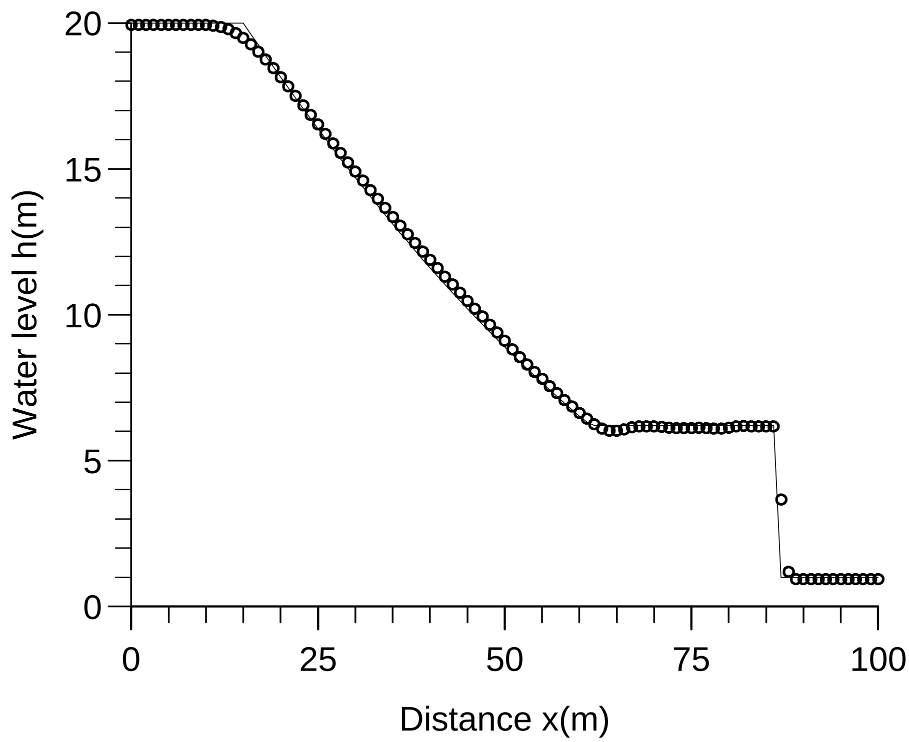
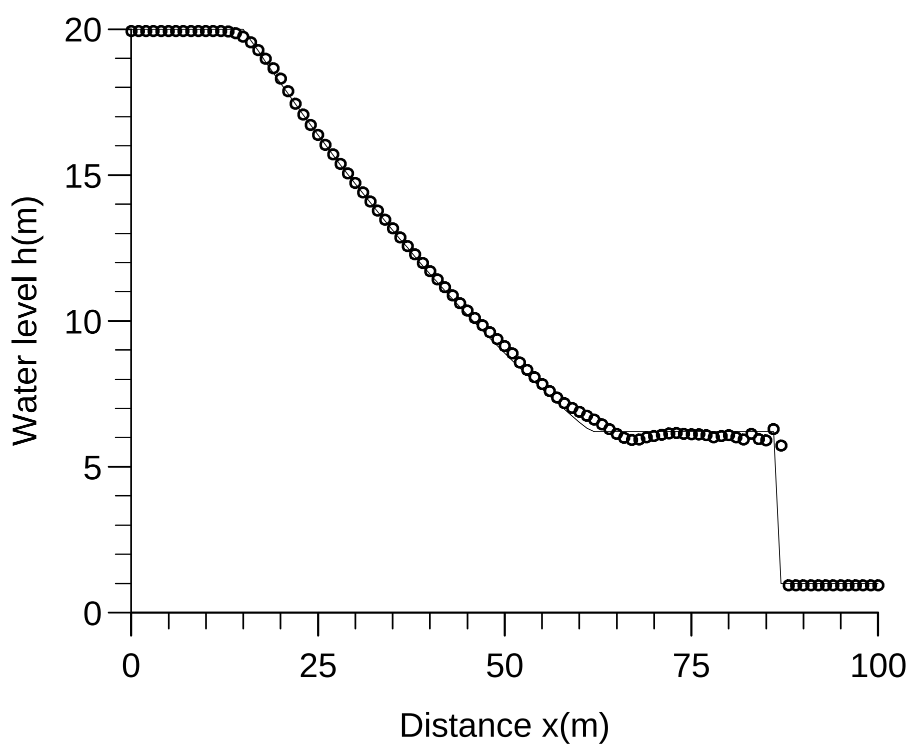


Figure 7: Recovery of Conservation: Cubic interpolation and Roe's scheme. N=101. Dambreak problem for a height ratio 5:1. Upper: CFL=0.75. Lower: CFL=1.75





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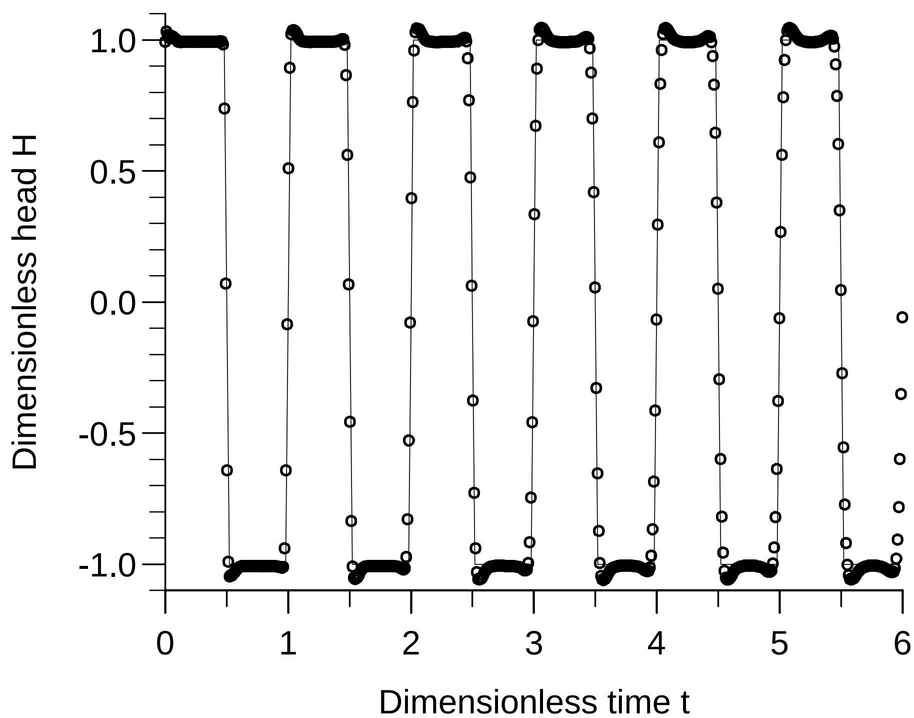
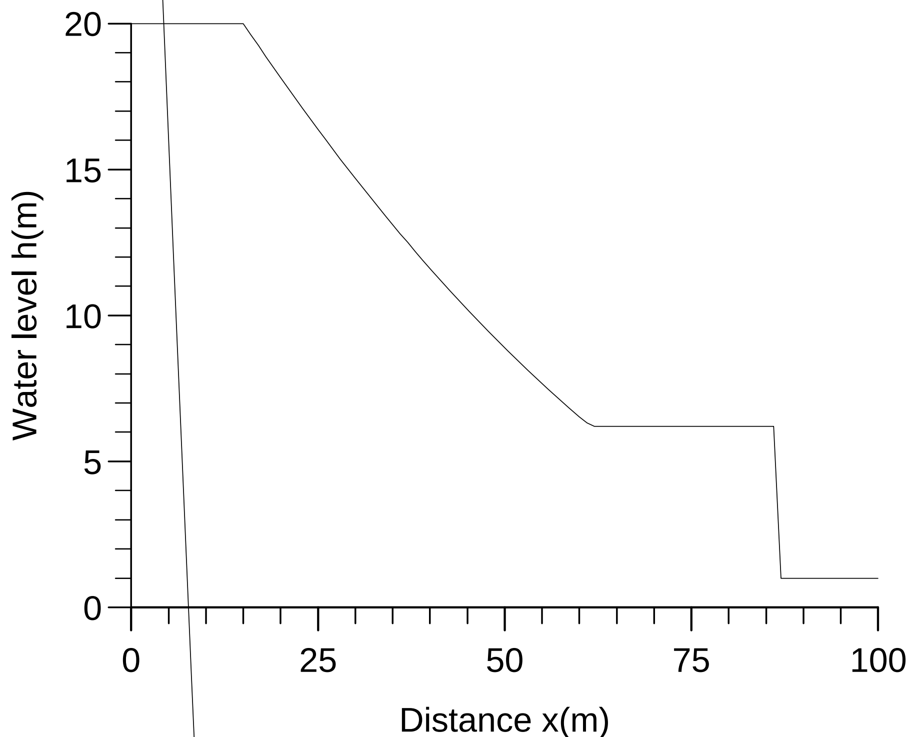


Dimensionless head H
1.0
0.5
0.0
-0.5
-1.0
0
1
2
3
4
5
6
Dimensionless time t


20
15
10
5
0
Water level h(m)
0
25
50
75
100
Distance x(m)