Existence and properties of solutions for neural eld equations

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Abstract

The rst goal of this work is to study solvability of the neural eld equation

$$\frac{@u(x;t)}{@t} \quad u(x;t) = \int_{R^m} w(x;y$$

networks in the 1940ies [1{8]. One particular neuron model with certain physiological significance is the *leaky integrator unit* [2, 3, 5{8] described by the ODEs

$$\frac{du_i(t)}{dt} + u_i(t) = \sum_{j=1}^{N} w_{ij}\, f(u_j(t))\,. \tag{1}$$

Here $u_i(t)$ denotes the time-dependent membrane potential of the ith neuron in a network of N units with synaptic weights w_{ij}. The nonlinear function f describes the conversion of the membrane potential $u_i(t)$ into a spike train $r_i(t) = f(u_i(t))$, and is called the *activation function*.

The left-hand-side of Eq.(1) describes the intrinsic dynamics of a leaky integrator unit, i.e. an exponential decay of membrane potential with time constant . The right-hand-side of Eq.(1) represents the *net-input* to unit i: the weighted sum of activity delivered by all units j that are connected to unit i ($j \to i$). Therefore, the weight matrix $W = (w_{ij})$ comprises three different kinds of information: (1) unit j is connected to unit i if $w_{ij} \neq 0$ (connectivity, network topology), (2) the *synapse* $j \to i$ is *excitatory* ($w_{ij} > 0$), or *inhibitory* ($w_{ij} < 0$), (3) the *strength* of the synapse is given by $|w_{ij}|$.

For the activation function f, essentially two different approaches are common. On the one hand, a *deterministic* McCulloch-Pitts neuron [1] is obtained from a Heaviside step function

$$f(s) := \begin{cases} 0; & s < \\ 1; & s \end{cases} \tag{2}$$

for $s \in \mathbb{R}$ with an *activation threshold* describing the *all-or-nothing-law* of action potential generation. Supplementing Eq.(1) with a resetting mechanism for the membrane potential, the Heaviside activation function provides a *leaky integrate and fire* neuron

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Starting with the leaky integrator network equation (1), the sum over all units is replaced by an integral transformation of a neural field quantity $u(x,t)$, where the continuous parameter $x \in \mathbb{R}^m$ now indicates the position i in the network. Correspondingly, the synaptic weight matrix w_{ij} turns into a kernel function $w(x,y)$. Then, Eq.(1) assumes the form of a *neural field equation* as discussed in [10,11]

$$\frac{\partial u(x,t)}{\partial t} - u(x,t) = \int_{\mathbb{R}^m} w(x,y) f(u(y,t))\, dy, \quad x \in \mathbb{R}^m, t > 0 \tag{4}$$

with *initial condition*

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^m. \tag{5}$$

Up to now, neural field equations have been investigated under serious restrictions upon the integral kernel $w(x,y)$, including homogeneity ($w(x,y) = w(x - y)$) and isotropy ($w(x,y) = w(|x - y|)$). In these cases, the technique of Green's functions allows the derivation of PDEs for the neural waves $u(x,t)$ assuming special kernels such as exponential, locally uniform or "Mexican hat" functions [13,14,18,23,26]. Solutions for such neural field equations have been obtained for macroscopic, stationary neurodynamics in order to predict spectra of the electroencephalogram (EEG) [14,17,19,22], or bimanual movement coordination patterns [12,13].

By contrast, heterogeneous kernels and thalamo-cortical loops in addition to homogeneous cortico-cortical connections have been discussed in [16] and [17,19,25], respectively. However, at present there is no universal neural field theory available, that would allow the study of field equations with general synaptic kernel functions. Yet such a theory would be mandatory for modeling mesoscopic and transient neurodynamics as is characteristic, e.g., for cognitive phenomena.

Our goal is hence to develop a mathematical theory of neural fields starting with the typical example of leaky integrator field equations. We expect that our analysis will serve as a model for various variations and generalizations of neural field equations which are currently being investigated for applications in the field of cognitive neurodynamics [27].

In this paper we shall examine the solvability of the *integro-differential equation* (4) with tools from functional analysis, the theory of ordinary differential equations and integral equations. We will provide a proof of global existence of solutions and study their properties in dependence on the smoothness of the synaptic kernel function w and the smoothness of the activation function f.

2 The neural field equation

For studying the existence of solutions of the neural field equation (4) we define the operator

$$(Fu)(x,t) := \frac{1}{\tau}\left(-u(x,t) + \int_{\mathbb{R}^m} w(x,y) f(u(y,t))\, dy\right), \quad x \in \mathbb{R}^m,\ t > 0. \tag{6}$$

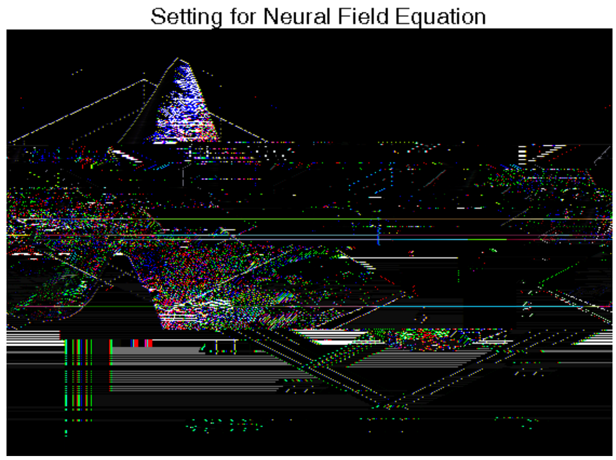


Figure 1: We show the setting for the neural field equation (4) for the case $m = 1$. The potential $u(x;t)$ is depending on space $x \in \mathbb{R}^m$ and time t 0. Here, a pulse is travelling in the x-direction when time increases. The plane indicates the cut-off parameter in the activation function f. Only a field $u(x;t)$ will contribute to the increase of the potential.

Then the neural field equation (4) can be reformulated as

$$u' = Fu; \tag{7}$$

where u' denotes the derivative of u with respect to the time variable t. For later use we also define the operators

$$(Au)(x;t) := \int_0^t (Fu)(x;s)\,ds; \quad x \in \mathbb{R}^m;\ t > 0; \tag{8}$$

and

$$(Ju)(x;t) := \frac{1}{\ } \int_{\mathbb{R}^m} w(x;y) f(u(y;t))\,dy; \quad x \in \mathbb{R}^m;\ t > 0: \tag{9}$$

To define appropriate spaces and study the mapping properties of the operators F and A we need to formulate conditions on the synaptic weight kernel w and the activation function f in the neural field equation. Here, we will study two classes of functions f.

The *first* class contains smooth functions f. In this case we can employ tools from the classical theory of ordinary differential equations to obtain existence results.

The *second* class works with non-smooth functions f, as for example when f is a Heaviside jump function. In this case the above theory is not applicable and we will construct counterexamples. We will study the existence problem by investigating particular kernels w which allow particular solutions.

2.1 General estimates for solutions to the NFE

Proof. We first note that the term Ju defined in (9) can be estimated by

$$(Ju)(x;t) \quad \frac{C_w}{} \tag{18}$$

Next, we observe that the derivative $u'(t)$ in the neural field equation is bounded by

$$u'(x;t) \quad bu(x;t) + c; \qquad u'(x;t) \quad bu \quad c \tag{19}$$

with $b = 1/$ and $c = C_w/$. Thus, the value of $u(t)$ will be bounded by the solution to the ordinary differential equation (77) with $a = u_0(x)$, $b = 1/$ and $c = C_w/$. According to Lemma 4.1 the bound is given by C_{tot} defined in (16). This proves the estimate (17).

2.2 The NFE with a smooth activation function f

Here, for the function $f : \mathbb{R} \to \mathbb{R}$ we assume that

$$f \in BC^1(\mathbb{R}); \tag{20}$$

With the conditions (10) to (14) we now obtain the following mapping properties of the neural field operator F.

LEMMA 2.3. *The operator F defined by (6) with kernel w and activation function f which satisfy the conditions of Definition 2.1 and (20) is a bounded nonlinear operator on $BC(\mathbb{R}^m) \quad C^1(\mathbb{R}_0^+)$*

Since $u(x,t)$ is continuous in x we obtain the continuity of Fu in x. Finally, we need to show that Fu is continuously differentiable with respect to the time variable. This is clear for the first term $-u(x,t)/\tau$. The time-dependence of the integral

$$(Ju)(x,t) := \int_{\mathbb{R}^m} w(x,y) f(u(y,t))\, dy \tag{22}$$

is implicitly given by the time-dependence of the field $u(y,t)$. By assumption we know that $u(x,\cdot) \in C^1(\mathbb{R}_0^+)$ and the function f is $BC^1(\mathbb{R})$. Then via the *chain rule* we derive

$$\frac{d}{dt} f(u(y,t)) = \frac{df(s)}{ds}\bigg|_{s=u(y,t)} \cdot \frac{\partial u(y,t)}{\partial t}.$$

Since f' is bounded on $\mathbb{R}$ and w is integrable we obtain the differentiability of the integral with the derivative

$$\frac{\partial Ju}{\partial t}(x,t) = \int_{\mathbb{R}^m} w(x,y) \left\{ \frac{df}{ds}(u(y,t)) \cdot \frac{\partial u}{\partial t}(y,t) \right\} dy, \quad t > 0. \tag{23}$$

The function $\partial Ju/\partial t(x,t)$ depends continuously on $t \in \mathbb{R}^+$ due to the continuity of df/ds and du/dt in t and the term (23) is bounded for $t \geq 0$ and $x \in \mathbb{R}^m$. This completes the proof.

By integration with respect to t we equivalently transform the neural field equation (4) or (7), respectively, into a *Volterra integral equation*

$$u(x,t) = u(x,0) + \int_{s=0}^{t} (Fu)(x,s)\, ds, \quad x \in \mathbb{R}^m,\ t > 0, \tag{24}$$

which, with A defined in (8), can be written in the form

$$u(x,t) = u(x,0) + (Au)(x,t), \quad x \in \mathbb{R}^m,\ t > 0. \tag{25}$$

Lemma 2.4. *The Volterra equation (24) or (25), respectively, is solvable on* $\mathbb{R}^m \times (0,\rho)$ *for some* $\rho > 0$ *if and only if the neural field equation (4) or (7), respectively, is solvable for* $x \in \mathbb{R}^m$ *and* $t \in (0,\rho)$. *In particular, solutions to the Volterra equation (24) are in* $BC^1(\mathbb{R}_0^+)$.

Proof. If the neural field equation is solvable with some continuous function $u(x,t)$, we obtain the Volterra integral equation (24) for the solution u by integration.

To show that a solution $u(x,t)$ to the Volterra integral equation (24) in $BC(\mathbb{R}^m) \times BC(\mathbb{R}_0^+)$ satisfies the neural field equation (4) we first need to ensure sufficient regularity, since solutions to equation (4) need to be differentiable with respect to t. We note that the function

$$g_x(t) := \int_0^t (Fu)(x,s)\, ds, \quad t > 0$$

is differentiable with respect to t with continuous derivative for each $x \in \mathbb{R}^m$. Thus, the solution $u(x,t)$ to equation (24) is continuously differentiable with respect to $t > 0$ and the derivative is continuous on $[0,\infty)$. Now, the derivation of (4) for u from (24) is straightforward by differentiation.

An important preparation for our local existence study is the following lemma. We need an appropriate local space, which for $\rho > 0$ is chosen as

$$X_\rho := BC(\mathbb{R}^m) \times BC([0,\rho]). \tag{26}$$

The space X_ρ equipped with the norm

$$\|u\|_\rho := \sup_{x \in \mathbb{R}^m, t \in [0,\rho]} |u(x,t)| \tag{27}$$

is a Banach space. For $\rho = \infty$ we denote this space by X, i.e.

$$\begin{aligned} X &:= BC(\mathbb{R}^m) \times BC(\mathbb{R}_0^+), \\ \|u\|_X &:= \sup_{x \in \mathbb{R}^m, t \in \mathbb{R}_0^+} |u(x,t)|. \end{aligned} \tag{28}$$

An operator $\tilde{A}$ from a normed space X into itself is called a *contraction*, if there is a constant q with $0 < q < 1$ such that

$$\|\tilde{A}u_1 - \tilde{A}u_2\| \le q\|u_1 - u_2\| \tag{29}$$

is satisfied for all $u_1, u_2 \in X$. A point $u^* \in X$ is called *fixed point* of $\tilde{A}$ if

$$u^* = \tilde{A}u^* \tag{30}$$

is satisfied. We are now prepared to study the properties of A on X_ρ.

Lemma 2.5. *For $\rho > 0$ chosen sufficiently small, the operator A is a* contraction *on the space X_ρ defined in (26).*

Proof. We estimate $Au_1 - Au_2$ and abbreviate $u := u_1 - u_2$. We decompose $A = A_1 + A_2$ into two parts with the linear operator

$$(A_1 v)(x,t) := \frac{1}{\tau}\int_0^t v(x,s)\, ds, \quad x \in \mathbb{R}^m, \ t > 0, \tag{31}$$

and the nonlinear operator

$$(A_2 v)(x,t) := \frac{1}{\tau}\int_0^t \int_{\mathbb{R}^m} w(x,y) f(v(y,s))\, dy\, ds, \quad x \in \mathbb{R}^m, \ t > 0. \tag{32}$$

We can estimate the norm of A_1 by

$$\|A_1 u\| \le \frac{\rho}{\tau}\|u\|_\infty; \tag{33}$$

which is a contraction if ρ is sufficiently small. Since $f \in BC^1(\mathbb{R})$ there is a constant L such that

$$|f(s) - f(\tilde{s})| \le L|s - \tilde{s}|, \quad s, \tilde{s} \in \mathbb{R}. \tag{34}$$

This yields

$$\begin{aligned} |Ju_1(x;t) - Ju_2(x;t)| &\le \frac{1}{\tau}\int_{\mathbb{R}^m} |w(x;y)|\,|f(u_1(y;t)) - f(u_2(y;t))|\,dy \\ &\le \frac{1}{\tau}L\int_{\mathbb{R}^m} |w(x;y)|\,|u_1(y;t) - u_2(y;t)|\,dy \\ &\le \frac{1}{\tau}LC_w\|u_1 - u_2\|_\infty. \end{aligned} \tag{35}$$

Finally, by an integration with respect to t we now obtain the estimate

$$\|A_2u_1 - A_2u_2\| \le \frac{\rho}{\tau}LC_w\|u_1 - u_2\|_\infty. \tag{36}$$

For ρ sufficiently small the operator A_2 is a contraction on the space X_ρ. For

$$q := \frac{\rho}{\tau}(1 + LC_w) < 1 \tag{37}$$

the operator $A = A_1 + A_2$ is a contraction on X_ρ.

Now, the local existence theorem is given by the following theorem.

THEOREM 2.6 (Local existence for NFE). *Assume that the synaptic weight kernel w and the activation function f satisfy the conditions of Definition 2.1 and (20) and let $\rho > 0$ be chosen such that (37) is satisfied with L being the Lipschitz constant of f. Then we obtain existence of solutions to the neural field equations on the interval* $[0;\rho]$.

Remark. The result is a type of Picard-Lindelöf theorem for the neural field equation (4) under the conditions of Definition 2.1 and (20).

Proof. We employ the Banach Fix-Point Theorem to the operator equation (25). We have shown that the operator A is a contraction on X_ρ defined in (26). Then, also the operator $\tilde{A}u := u_0 + Au$ is a contraction on the complete normed space X_ρ. Now, according to the Banach fixpoint theorem the equation

$$u = \tilde{A}u \tag{38}$$

as a short form of the Volterra equations (25) or (24), respectively, has one and only one fixpoint u . This proves the unique solvability of (24). Finally, by the equivalence Lemma 2.4 we obtain the unique solvability of the neural field equation (4) on $t \in [0; \ ]$.

In a last part of this section we combine the global estimates with local existence to obtain a global existence result.

THEOREM 2.7 (Global existence of solutions to NFE). *Under the conditions of Definition 2.1 we obtain existence of global bounded solutions to the neural field equation.*

Proof. We first remark that the neural field equation does not explicitly depend on time. As a result we can apply the local existence result with the same constant to any interval $[t_0; t_0 + \ ] \ \mathbb{R}$ when initial conditions $u(x; t_0) = u_0$ for $t = t_0$ are given. This means we can use Theorem 2.6 iteratively.

First, we obtain existence of a solution on an interval $I_0 := [0; \ ]$ for

$$:= \frac{\quad}{2(1 + LC_w)}: \tag{39}$$

Then, the function $u_1(x) := u(x; \ )$ serves as new initial condition for the neural field equation on $t > \ $ with initial conditions u_1 at $t = \ $. We again apply Theorem 2.6 to this equation to obtain existence of a solution on the interval $I_1 = [\ ; 2\ ]$.

This process is continued to obtain existence on the intervals $I_n := [n\ ; (n + 1)\ ]$, $n \in \mathbb{N}$, which shows existence for all $t \in \mathbb{R}$. Global bound for this solution have been derived in Lemma 2.2.

2.3 The NFE with a Heaviside activation function f

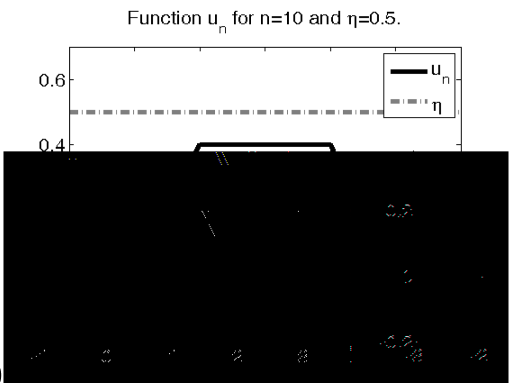
In this section we will construct special solutions to the neural field equation in the case of an activation function f given by Eq.(2). In this case the results of the preceding sections are no longer applicable. We will develop specific methods to analyse the solvability of the equation for this particular case.

We first show that for the activation function f defined in (2), the operator F does not longer depend continuously on the function u.

LEMMA 2.8. *With f given by (2), w according to Definition 2.1 and the additional condition (15) for the kernel the function Fu does not depend continuously on $u \in X$ with X defined in (28).*

Proof. Consider the sequence $(u_n)_{n\in\mathbb{N}}$ of functions $u_n \in X$ with

$$u_n(x; t) := \begin{cases} 0; & x \quad 2 \\ (\quad \frac{1}{n}) \ (2 + x) & x \in (\ 2; \ 1) \\ \frac{1}{n}; & x \in [\ 1; 1] \\ (\quad \frac{1}{n}) \ (2 \quad x) & x \in (1; 2) \\ 0; & x \quad 2; \end{cases} \tag{40}$$



(a)

Figure 2: In (a) we show a function u_n which is used to prove the non-continuity of the operator F for a Heaviside-type activation function f in the neural field equation.

for $x \in \mathbb{R}$ and $t \geq 0$, compare Figure 2. The function u is defined by (40) with $n = \infty$, where we use $1/\infty = 0$. Then $u_n \to u$ for $n \to \infty$ in X. For all $n \in \mathbb{N}$ we have $Fu_n = -u_n/\tau$, since $f(u_n(y;t)) = 0$ for $y \in \mathbb{R}^m$ and $t \geq 0$. However, we calculate

$$(Fu)(x;t) = -\frac{1}{\tau}u(x;t) + \frac{1}{\tau}\underbrace{\int_{[-1;1]} w(x;y)dy}_{=:J(x)}. \tag{41}$$

Thus, we have

$$\lim_{n\to\infty}\left(Fu_n(x;t) - Fu(x;t)\right) = J(x), \quad x \in \mathbb{R}, \tag{42}$$

i.e. for general kernels $w(x;y)$ where $J(x) \not\equiv 0$ the operator F is not continuous.

Remark. As a consequence of Lemma 2.8 the operator A is not a contraction on X for any $\rho > 0$, since

$$\begin{aligned} Au_n(t) - Au(t) &= \frac{1}{\tau}\left(-\int_0^t \left(u_n(x;s) - u(x;s)\right) ds \right. \\ &\quad \left. + \int_0^t \int_{\mathbb{R}} w(x;y)\left(f(u_n(y;s)) - f(u(y;s))\right) dy\, ds\right) \\ &\to -J(x)\, t, \quad n \to \infty, \end{aligned} \tag{43}$$

where $J(x)$ is given by (41).

Since the operator A does not depend Lipschitz continuously on u, we need to use techniques different from the Banach fixpoint theorem above. Here, we will develop an approach based on compactness arguments to carry over the existence results from above to the non-smooth Heaviside activation function f. To this end we define the Hölder space

$$X_{\alpha,\tau} := BC^{\alpha}(\mathbb{R}^m) \otimes BC^{\alpha}([0,\tau]) \tag{44}$$

for $\alpha \in (0,1]$ equipped with the Hölder norm

$$\begin{aligned} \|\varphi\|_{\alpha,\tau} := \|\varphi\|_{\infty} &+ \sup_{t\in[0,\tau],x,y\in\mathbb{R}^m} \frac{|\varphi(x,t) - \varphi(y,t)|}{|x-y|^{\alpha}} \\ &+ \sup_{x\in\mathbb{R}^m,t,s\in[0,\tau]} \frac{|\varphi(x,t) - \varphi(x,s)|}{|t-s|^{\alpha}} \end{aligned} \tag{45}$$

It is well known that the Hölder space on a compact set M is compactly embedded into the space $BC(M)$. However, for unbounded sets like the space $\mathbb{R}^m$ this is not the case. However, we still get *local compactness* of the embedding, i.e. every bounded sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $X_{\alpha,\tau}$ does have a subsequence $(\tilde{\varphi}_k)_{k\in\mathbb{N}}$ which is *locally* converging in X_{τ} towards an element $\varphi \in X_{\tau}$, i.e. where

$$\sup_{t\in[0,\tau];x\in B_R(0)} \left|\tilde{\varphi}_k(x,t) - \varphi(x,t)\right| \to 0, \quad n \to \infty \tag{46}$$

for every fixed $R > 0$. We need some of the mapping properties of the operators A_1 and A_2 defined in (31) and (32), respectively, in these spaces. This is the purpose of the following lemma. Define the *indicator function* of a set M by

$$\chi_M(x) := \begin{cases} 1, & x \in M \\ 0, & x \notin M. \end{cases} \tag{47}$$

Lemma 2.9. *The operator A_1 is a linear operator which maps X_{τ} boundedly into X_{τ} with norm bounded by τ/η. In particular, for $\tau < \eta$ the operator $I - A_1$ is invertible on X_{τ} with bounded inverse given by*

$$(I - A_1)^{-1} = \sum_{l=0}^{\infty} A_1^l \tag{48}$$

Moreover, the operators A_1, $I - A_1$ and $(I - A_1)^{-1}$ are local *with respect to the variable x with local bounds in the sense that*

$$A_1(\chi_M u) = (\chi_M A_1)(u), \quad u \in X_{\tau}, \tag{49}$$

for all open sets $M \subset \mathbb{R}^m$ where $\chi_M A_1$ is bounded in $BC(M) \otimes BC([0,\tau])$ by τ/η. These operators map a locally convergent sequence onto a locally convergent sequence.

Proof. The linearity of A_1 is trivial and the bound of the operator A_1 has been derived in (33). Then the form (48) is the classical Neumann series in normed spaces. Clearly, the operator A_1 and $I \quad A_1$ are local in x in the sense of (49). And the bound $\ = \ $ holds for $_M \ A_1$.

Consider a bounded locally convergent sequence $(\ _n)_{n \in \mathbb{N}} \quad X$. Then we have

$$A_1(\ _n \quad)(x; t) \qquad \frac{1}{\ } \int_0^t \ _n(x; s) \quad (x; s) \ ds \to 0; \quad n \to \infty; \tag{50}$$

uniformly for $x \in B_R(0)$ and $t \in [0; \ ]$ for each xed $R > 0$. This means that $A_1 \ _n$ is a locally convergent sequence. The same arguments apply to $I \quad A_1$ and $(I \quad A_1)^{\ 1}$, and the proof is complete.

We have seen above that the operator F is not continuous on X or X , respectively. The same is true for the operator A_2. However, we will see that the operators are bounded

We consider a sequence of nonlinear smooth functions $f_n : \mathbb{R} \to [0,1]$ such that

$$f_n(t) = 0 \text{ on } [-\infty, \eta - \tfrac{1}{n}]; \quad f_n(t) = 1 \text{ on } [\eta, \infty). \tag{53}$$

Such a sequence can be easily constructed with arbitrary degree of smoothness. We will denote the operators depending on the nonlinearity functions f_n by A_n and F_n and the operators with the function f by A and F, respectively. We split the operator A_n into $A_n = A_1 + A_{2,n}$. The operator A_2 with the discontinuity in the nonlinearity f generates some difficulties, which are reflected by the following result.

LEMMA 2.11. *For fixed $u \in X$ we have $A_{2,n}u \to A_2 u$ locally. The convergence does not hold in the operator norm.*

Proof. We estimate

$$\begin{aligned}
\delta_n &:= \left| A_2 u(x,t) - A_{2,n}u(x,t) \right| \\
&\leq \left| \int_0^t \int_{\mathbb{R}^m} w(x,y)\big(f(u(y,t)) - f_n(u(y,t)) \big)\, dy\, ds \right| \\
&\leq \int_0^t \int_{\mathbb{R}^m} |w(x,y)| \left| f(u(y,t)) - f_n(u(y,t)) \right| dy\, ds
\end{aligned}$$

Now with $M_n(t) := \{ y \in \mathbb{R}^m : u(y,t) \in \operatorname{supp}(f - f_n) \}$ we estimate this by

$$\delta_n \leq \int_0^t \int_{M_n(t)} |w(x,y)|\, dy\, ds \to 0, \quad n \to \infty \tag{54}$$

as a result of (53). This holds uniformly on compact sets, but in general it does not hold uniformly for $x \in \mathbb{R}^m$.

For some function $v \in X$ we define the set

$$M_{\eta,\tau,R}[v] := \left\{ (y,s) \in \overline{B_R(0)} \times [0,\tau] : v(y,s) = \eta \right\}, \tag{55}$$

i.e. $M_{\eta,\tau,R}[v]$ is the set of space-time points (y,s) in $B_R(0) \times [0,\tau]$ where $v(y,s)$ equals the threshold η in the Heaviside nonlinearity. When we use $R = \infty$ then in this definition $B_\infty(0)$ is equal to $\mathbb{R}^m$. By $\mu(M)$ we denote the Euclidean area, volume or more general *Euclidean measure*

$$\mu(M) := \int_M 1\, dy \tag{56}$$

of a set M. We call an operator A_2 locally continuous if for a locally convergent sequence $u_n \to u$ we have $A_2 u_n \to A_2 u$.

LEMMA 2.12. *The operator A_2 is locally continuous in $v \in X$ if and only if the volume of $M_{\,;\,;\infty}[v]$ is zero. Moreover, in this case we have*

(57) $$u_n \xrightarrow{loc} u \;\Rightarrow\; A_{2;n}u_n \xrightarrow{loc} A_{2;n}u.$$

Proof. We need to start with some preparations. We first note that when $(M_{\,;\,;\infty}[v])$ is zero this is the case also for all $M_{\,;\,;R}[v]$ with $R > 0$. The set $M_{\,;\,;R}[v]$ is a closed set, thus $B_R(0) \setminus M_{\,;\,;R}[v]$ is an open set. We choose a sequence G_l, $l \in \mathbb{N}$ of closed sets $G_l \subset B_R(0) \setminus M_{\,;\,;R}[v]$ such that

$$_l := (B_R(0) \setminus G_l) \to 0, \quad l \to \infty.$$

Second, if $v_n \to v$ locally in X, then for each $l \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $f(v_n(y;s)) = f(v(y;s))$, $(y;s) \in G_l$, for all $n \geq N$.

We are now prepared to prove continuity of A_2 in v. Let v be given with $(M_{\,;\,;\infty}[v]) = 0$ and $(v_n)_{n\in\mathbb{N}}$ be a sequence in X with $v_n \to v$ locally. Given some $r > 0$ and > 0 we proceed as follows.

(1) We choose $R > 0$ such that

$$\int_{\mathbb{R}^m \setminus B_R(0)} |w(x;y)|\, dy \leq \frac{}{2}; \quad x \in B_r(0).$$

The existence of such R is a consequence of the condition $w(x;\cdot) \in L^1(\mathbb{R}^m)$ which is continuous in $x \in \mathbb{R}^m$ and bounded on the compact set $\overline{B_r(0)}$.

(2) On $B_R(0)$ we choose $L \in \mathbb{N}$ such that

$$_L\, C_1 \leq \frac{}{2}.$$

(3) Given L we choose N sufficiently large such that on G_L we have

(58) $$f(v_n(y;s)) = f(v(y;s)); \quad (y;s) \in G_L$$

for all $n \geq N$.

We now estimate the integral

(59) $$A_2 v_n(x;t) - A_2 v(x;t) \leq \int_0^t \int_{\mathbb{R}^m} w(x;y) \; f(v_n(y;s)) - f(v(y;s)) \; dy\, ds$$

by a decomposition of the integration over $\mathbb{R}^m$ into one over

$$M_1 := \mathbb{R}^m \setminus B_R(0); \quad M_2 := B_R(0) \setminus G_L; \quad M_3 := G_L.$$

The three integrals can be estimated by (1), (2) and (3) and we obtain

(60) $$A_2 v_n(x;t) - A_2 v(x;t) \leq ; \quad x \in B_r(0); t \in [0;]; \quad n \geq N().$$

This shows local continuity of A_2 in v.

If the volume of $M(v)$ is not zero, there is a set $G \subset \mathbb{R}^m \times [0;]$ with $(G) > 0$ where $v(y;s) = $. In this case as in Lemma 2.8 we can construct a sequence of functions $v_n \in X$ which converges to v such that they are equal to v on $\mathbb{R}^m \times [0;] \setminus G$ and $v_n(y;s) <$ on the open interior of G. In this case we obtain a remainder term

$$A_2 v_n(x;t) - A_2 v(x;t) \to \int_G w(x;y)\, dy\, ds > 0; \quad n \to \infty;$$

according to (15). This proves that in this case the operator A_2 is not continuous in v. The more general convergence (57) is shown with the same arguments, where the equality (58) needs to be replaced by some estimate involving f_n.

We will now carry out the basic steps to study solvability of the discontinuous equation.

We consider solutions $u_n \in X$ for some with $= < 1$ of the Volterra equation (24) with function f_n for $n \in \mathbb{N}$, i.e.

$$u_n - Au_n = u_0; \quad n \in \mathbb{N}. \tag{61}$$

Then, the operator $I - A_1$ is linear and invertible in X . Multiplication by the operator $(I - A_1)^{-1}$ leads to the equivalent equation

$$u_n - (I - A_1)^{-1} A_{2;n} u_n = (I - A_1)^{-1} u_0; \quad n \in \mathbb{N}. \tag{62}$$

According to Lemma 2.2, the sequence $(u_n)_{n \in \mathbb{N}}$ of (4) on $[0;]$ is bounded uniformly by the constant C_{tot} in X . Then, the sequence

$$_n := A_{2;n} u_n; \quad n \in \mathbb{N}; \tag{63}$$

is bounded in $X_{;}$ for > 0. By the locally compact embedding of $X_{;}$ into X , the sequence $(_n)_{n \in \mathbb{N}}$ has a locally convergent subsequence in X which we denote by $(_k)_{k \in \mathbb{N}}$ and its limit in X by . The operator $(I - A_1)^{-1}$ maps locally convergent sequences onto locally convergent sequences, thus the sequence

$$u_k = (I - A_1)^{-1} u_0 + (I - A_1)^{-1} A_{2;k} u_k; \quad k \in \mathbb{N}$$

is locally convergent towards some function u . In this case by application of $I - A_1$ we obtain

$$u + A_1 u = u_0.$$

THEOREM 2.13 (Local existence for Heaviside type activation function f). *Consider a kernel w which satisfies the conditions of Definition 2.1 with a Heaviside type activation function f given in (2) where we assume that $w \in BC^{0,}(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$. If an accumulation point u of solutions of $u_n - A_n u_n = u_0$ satisfies $(M_{;;1}(u)) = 0$, then u solves the equation $(I - A)u = u_0$, i.e. the Volterra integral equation (24) has a solution in X.*

We are now prepared to derive a global existence result with the same technique as in the previous section.

THEOREM 2.14 (Global existence for Heaviside type activation function f). *Consider a kernel w which satisfies the conditions of Definition 2.1 with a Heaviside type activation function f given in (2) where we assume that $w \in BC^{0,}(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$. If an accumulation point u of solutions of $u_n - A_n u_n = u_0$ satisfies $(M_{;1;1}(u)) = 0$, then the neural field equation (4) has a global solution for $t > 0$.*

3 Velocity and durability of neural waves

The goal of this part is to estimate the velocity and durability of neural waves. Here, we will say that a wave field *is relevant* at a point $x \in \mathbb{R}^m$ at time $t > 0$ if

$$u(x;t) \quad . \tag{64}$$

Otherwise a field is called *irrelevant* in x. The condition (64) arises in connection with the integral Ju given by (9) in (4), where local contributions from $x \in \mathbb{R}^m$ are given only if $u(x;t)$. We will consider the time in which fields which are zero in some part of the space reach a relevant magnitude or amplitude, respectively.

Speed estimates for a neural wave. To evaluate the maximal speed in space of a neural wave we must first define an appropriate setup for the *wave speed*. In our current model setup (4) with a non-local kernel $w(x;y)$ some field $u(x;t)$

The maximal speed of waves for the neural field equation (4) is given by

$$V_{\max} := \sup_{u_0 \text{ admisible}; x \in \mathbb{R}^m} \tag{66}$$

with some constant c_m depending on the dimension m. For the next steps we will directly work with a bound (70).

On $\mathbb{R}^m \setminus M$ the field was zero at $t = 0$. The local behavior of the field is bounded from above by

$$u(x,t) = \frac{c_m c}{(1 + d(x,M))^s} \quad (1 \quad e^{\;t/\;}), \quad x \in \mathbb{R}^m \setminus M, t \quad 0. \tag{71}$$

After some time T the supremum of the field u on $\mathbb{R}^m \setminus M$ will reach the threshold , i.e. $= c_m c(1 \quad e^{\;T/\;})$. We note that the derivative of this field at the boundary ∂M can be estimated via

$$\frac{d}{dr}\frac{\;}{(1 + r)^s}\Big|_{r=0} = \quad s \frac{1}{(1 + r)^{s+1}}\Big|_{r=0} = \quad s\;. \tag{72}$$

Let the boundary ∂M be located at $x = 0$ and consider only the one-dimensional case. The field $u(x,T + t)$ for $t = 0$ has a tangent $g(x) = \quad s\; x$ in $x = 0$. The curve has time derivative at $x = 0$ bounded by $u' = \quad + C_w$. Now, we can estimate the speed of the arguments x of $u(x,t) =$ defined in (71) by

$$u(x,T) + u'(x,T)\; t \qquad s\; x + (\quad + C_w)t \stackrel{!}{=}$$

which yields $x/t = (C_w \quad)/(\; s)$ and thus (69). This is a local estimate, but the front with $u(x,t) =$ will move along with the local speed and the above case is an upper estimate for any x and t. This completes the proof.

Remark. The speed estimate reflects important properties of the neural field equation. If the threshold approaches the maximal forcing term C_w, then the speed will be arbitrarily slow since the fields need more and more time to reach the threshold. If the decay exponent s increases, the speed becomes smaller. If the threshold is small, then the speed will be large. For $\; \to 0$ the speed diverges.

Durability of directed waves. We call a synaptic weight kernel w of the neural field equation *directed* if there is a direction $d_0 \in \mathbb{S}$ such that

$$w(x,y) \quad 0 \quad \text{for all } (x \quad y) \; d \quad 0. \tag{73}$$

Directedness of a kernel means that its influence to increase a field in some part of space

where we assume that $a < c{=}b$.

Uniqueness of solutions. First, we investigate uniqueness of the equation. Let $u_1; u_2$ be solutions and de ne $u = u_1 \quad u_2$. Then u solves the *homogeneous equation* $u^0 = \quad bu$ with $u(0) = 0$. Assume that there is some $t > 0$ such that $u(t) \neq 0$. Then we nd 0 such that $u(t) = 0$ for t 2 [0;] and $u(t) \neq 0$ for t 2 (;). Then, we divide by $u(t)$ to obtain

$$\frac{u^0(t)}{u(t)} = \quad b$$

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Matching the boundary condition $u(0) = a$ for $a < c/b$ yields

(81) $$a = c/b \quad e$$

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