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Preconditioners for inhomogenous anisotropic problems in spherical geometry

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Abstract

Inhomogenous Anisotropic Elliptic operators arise from the integration of the Navier-Stokes equations for a hydrostatic Boussinesq fluid on a sphere. Anisotropy may be defined as the variation of the property of a material with the direction in which it is measured. An Anisotropic Elliptic operator arises in the free-surface formulation of the Met O□ce Ocean model. A modified Helmholtz problem is iteratively solved using conjugate gradient with a diagonal preconditioner. The anisotropy in the corresponding discretised equations causes the convergence of the method to be slow, particularly in polar regions. Block diagonal and Alternating Direction Implicit preconditioners are considered here as alternatives and their impact on the pole problem and on the overall convergence are assessed.

1 Introduction

Most ocean models in use today are based on integrating the incompressible Navier-Stokes equations on a sphere. Complex topography is used at the ocean bottom and the ocean surface is either fixed or free to move with time. The ocean basins themselves typically contain irregularly shaped coastlines and islands which require the inclusion of specific boundary conditions into any solution algorithm.

The forms of the operators that arise in the spherical coordinate framework are anisotropic. An operator is anisotropic if its local properties vary with direction. As an example consider the constant coe□cient partial di□erential equation

$$i\frac{\partial}{\partial x}\left(L_x\frac{\partial U}{\partial x}\right)\; i\;\frac{\partial}{\partial y}\left(L_y\frac{\partial U}{\partial y}\right) = \circ(x;y$$

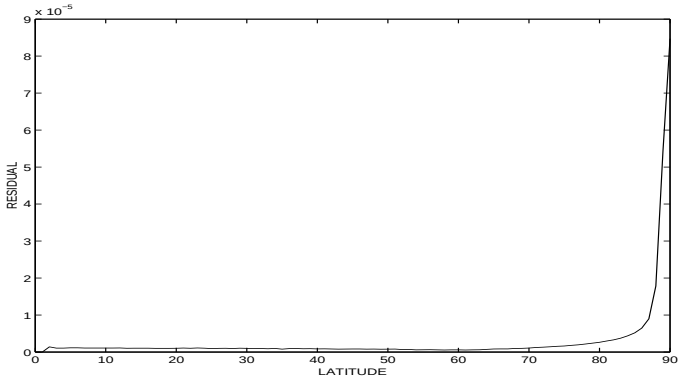

Figure 1: Latitudinal variance in convergence for experiment with free surface formulation of Met.O□ce ocean model (Northern Hemisphere only)

2 Met O□ce Free-surface Model

Most of the Ocean General Circulation models in use today, including the free-surface barotropic model used by the Met O□ce, are based on the Bryan-Cox-Semtner (henceforth BCS) model initially introduced by Bryan [2] in the late 1960's and later modified by Cox [3] and Semtner [9]. The BCS model solves the primitive equations, derived from the Navier-Stokes equations, in a spherical coordinate system using hydrostatic and Boussineq approximations. The implicit free-surface barotropic model was introduced by Dukowicz [4] and is summarised briefly here.
The barotropic, or vertically averaged, equations of state are given by

$$\begin{aligned} &\tfrac{@u}{@t} - fv = -g\tfrac{1}{a\cos\acute{A}}\tfrac{@\acute{}}{@¸} + G^x; \\ &\tfrac{@v}{@t} + fu = -g\tfrac{1}{a}\tfrac{@\acute{}}{@\acute{A}} + G^y; \\ &\tfrac{@\acute{}}{@t} + \tfrac{1}{a\cos\acute{A}}\left[\tfrac{@Hu}{@¸} + \tfrac{@Hv\cos\acute{A}}{@\acute{A}}\right] = 0; \end{aligned} \tag{2}$$

where $¸$ and $\acute{A}$ are longitude and latitude respectively, f is the coriolis parameter, g is the gravitational acceleration constant, $H = H(¸;\acute{A})$ is the total depth of the ocean, $(u;v)$ are the barotropic velocity components and G^x, G^y represent baroclinic forcing. Dukowicz [4] considered the following general time discretisation of equation (2) :

$$\begin{aligned} &\tfrac{u^{n+1}-u^{n-1}}{2¿} - fv^{®'} = -g\tfrac{1}{a\cos\acute{A}}\tfrac{@\acute{}^{®}}{@¸} + G^{x;n}; \\ &\tfrac{v^{n+1}-v^{n-1}}{2¿} + fu^{®'} = -g\tfrac{1}{a}\tfrac{@\acute{}^{®}}{@\acute{A}} + G^{y;n}; \\ &\tfrac{\acute{}^{n+1}-\acute{}^{n}}{¿} + \tfrac{1}{a\cos\acute{A}}\left[\tfrac{@Hu^{\mu}}{@¸} + \tfrac{@Hv^{\mu}\cos\acute{A}}{@\acute{A}}\right] = 0; \end{aligned} \tag{3}$$

with

$$\begin{aligned} u^{®'} &= ®^{0}u^{n+1} + (1-®^{0}-°^{0})u^{n} + °^{0}u^{n-1}; \\ \acute{}^{®} &= ®\acute{}^{n+1} + (1-®-°)\acute{}^{n} + °\acute{}^{n-1}; \\ u^{\mu} &= \mu u^{n+1} + (1-\mu)u^{n}: \end{aligned} \tag{4}$$

where ¿ is the fixed timestep, n is the current time level and ®, $®^{0}$, °, $°^{0}$ and μ are coe□cients used to parameterise the time centering of the pressure gradient, Coriolis, and divergence terms. Eliminating u^{n+1} and v^{n+1} in (3) and rearranging we can obtain an implicit equation for $\acute{}^{0}$ which represents the change in free surface height $\acute{}$ between two consecutive timesteps of the overall Met O□ce Unified model. The elliptic operator, which is solved at every timestep, is given by:

$$\frac{1}{a\cos\acute{A}}\left[\frac{@}{@¸}\left(\frac{H}{a\cos\acute{A}}\frac{@\acute{}^{0}}{@¸}\right) + \frac{@}{@\acute{A}}\left(\frac{H\cos\acute{A}}{a}\frac{@\acute{}^{0}}{@\acute{A}}\right)\right] - \acute{}^{0} = S(¸;\acute{A}); \tag{5}$$

where

$$- = \frac{1}{2\Re\mu g_{\dot{\iota}}}; \tag{6}$$

3 Preconditioners for model problem

We consider a Limited Area, Northern Hemisphere model problem of the following form in our numerical experiments:

$$\begin{cases} \frac{1}{\cos\acute{A}}\frac{\partial}{\partial \lambda}\left(\frac{1}{\cos\acute{A}}\frac{\partial U}{\partial \lambda}\right) + \frac{\partial}{\partial \acute{A}}\left(\cos\acute{A}\frac{\partial U}{\partial \acute{A}}\right) - kU = \circ(\lambda,\acute{A}) \\ \lambda \in (0^oE, 30^oE), \quad \acute{A} \in (10^oN, l) \\ U(0^oE,\acute{A}) = 0, U(30^oE,\acute{A}) = 0 \\ U(\lambda, 10^oN) = 0, U(\lambda, l) = 0 \\ l \in (40^oN, 89.5^oN). \end{cases} \tag{7}$$

where $k \geq 0$, and $\circ$ is known. In our experiments we investigated the effects of moving the northern boundary of the domain closer to the pole. We discretise the problem using a standard five-point discretisation scheme with a constant stepsize h in both directions and taking a natural ordering of the grid points. This gives rise to a matrix equation of the general form

$$AU = b, \tag{8}$$

where the variable U is a (unknown) column vector of the grid point values of the variable U and b is a (known) column vector representing boundary values and source terms. The system matrix A is a real, symmetric, $m \times m$ matrix representing the discretised model equations (where m is the number of grid points). It is also square, sparse, irreducible and diagonally dominant with strict diagonal dominance in at least one row. It is therefore irreducibly diagonally dominant and hence positive-definite ([10]).

The Met O□398(in1rc7cree-surfa8(in1r3o)-27(del)]Tn1r3ocuedu27(t)-)-331rc7usdel311rc7thirndi7(t)-ntA

4 Numerical Experiments

Figure 2 shows the effect the increased anisotropy due to moving the northern boundary closer to the pole has on the eigenvalues of G_D, the point Jacobi iteration matrix. We observe the clustering of secondary eigenvalues of G_D near the, slightly larger, leading eigenvalue. This suggests that more eigenmodes will contribute significantly to the errors with increased anisotropy. Figure 3 shows the leading four eigenvectors of G_D. Whilst the lead eigenmode does not possess a significant signal in the polar regions the others do and it is these that become more significant with increased anisotropy and therefore will contribute much more to the residual errors in the method.

Tables 1 and 2 show the effect on the conditioning of the problem with the increased anisotropy due to moving the northern boundary closer to the pole, and the use of the different preconditioners (where G_P is the iteration matrix of the preconditioned system $P^{-1}A$ with $G_P = I - P^{-1}A$). We observe that the conditioning becomes over an order of magnitude larger by moving the boundary near to the pole. We also observe that the conditioning of the matrix with the Block preconditioner is better than with the

q16Tf.4w109.9TD[42-156.15290.4w6JE47.97Diagonal09.9-142.41m289.45-6Tf.4w289.45-142.413.

	$\cdot(A)$		
Boundary	$h = \frac{1}{2}^o$	$h = 1^o$	$h = 2^o$
40^o	$2.19 £ 10^3$	544.18	134.17
70^o	$4.28 £ 10^3$	$1.04 £ 10^3$	249.57
88^o	$3.12 £ 10^4$	$6.51 £ 10^3$	$1.21 £ 10^3$
89^o	$5.20 £ 10^4$	$9.75 £ 10^3$	NA

Table 1: Variation of condition number with varying northern boundary, $k = 0.01$.

Preconditioner	$\cdot(P^{-1}A)$	$\rho(G_P)$
None		

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References

[1] Axelsson, O. *Iterative Solution Methods.* Cambridge University Press, 654pp.

[2] Bryan, K. *A numerical method for the study of the circulation of the World Ocean.* Journal of Computational Physics, 4, 347-376 (1969)

[3] Cox, M.D. *A primitive equation, three-dimensional model of the ocean.* GFDL Ocean Group Technical Report, 1 (1984)

[4] Dukowicz, J.K., Smith, R.D. *Implicit free-surface method for the Bryan-Cox-Semtner Ocean Model.* Journal of Geophysical Research, 99, 7991-8014 (1994)

[5] Feingold, D.G., Varga, R. *Block Diagonally Dominant matrices and Generalizations of the Gerschgorin theorem.* Pacific Journal of Mathematics, 12, 1241-1250 (1962)

[6] Golub, G.H., Van Loan, C.F. *Matrix Computations.* North Oxford Academic Press, 476pp. (1983)

[7] Hackbusch, W. *Iterative Solution of Large Sparse Systems of Equations.* Springer-Verlag, 432pp. (1994)

[8] Nichols, N.K. *Numerical solution of elliptic di erential equations.* PHd Thesis (1969)

[9] Semtner, A.J. *Finite-di erence formulation of a world ocean model* Advanced Physical Oceanographic Numerical Modelling, edited by J.J.O'Brien, 187-202 (1986)

[10] Varga, R.S. *Matrix Iterative Analysis* Prentice Hall, 322pp. (1962)