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Abstract

The problem of finding good preconditioners for the numerical solution of a certain important class of indefinite linear systems is considered. These systems are of a 2 by 2 block (KKT) structure in which the (2,2) block (denoted by $-C$) is assumed to be nonzero.

In *Constraint preconditioning for indefinite linear systems,* SIAM J. Matrix Anal. Appl., 21 (2000), Keller, Gould and Wathen introduced the idea of using constraint preconditioners that have a specific 2 by 2 block structure for the case of C being zero. We shall give results concerning the spectrum and form of the eigenvectors when a preconditioner of the form considered by Keller, Gould and Wathen is used but the system we wish to solve may have $C \neq 0.$ In particular, the results presented here indicate clustering of eigenvalues and, hence, faster convergence of Krylov subspace iterative methods when the entries of C are small; such a situations arise naturally in interior point methods for optimization and we present results for such problems which validate our conclusions.

1 Introduction

The solution of systems of the form

$$\underbrace{\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix}}_{\mathcal{A}_C} \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} c \\ d \end{bmatrix}}_{b}, \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{m \times m}$ are symmetric and $B \in \mathbb{R}^{m \times n}$, is often required in optimization and other various fields, Section 1.1. We shall assume that $0 < m \leq n$ and B is of full rank. Various preconditioners which take the general form

$$P_c = \begin{bmatrix} G & B^T \\ B & -C \end{bmatrix}, \tag{2}$$

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where $G \in \mathbb{R}^{n \times n}$ is some symmetric matrix, have been considered (for example, see [3, 4, 5, 8, 18, 23].) When $C = 0$, (2) is commonly known as a constraint preconditioner [2, 16, 17, 19]. In practice C is often positive semi-definite (and frequently diagonal).

As we will observe in Section 1.1, in interior point methods for constrained optimization a sequence of such problems are solved with the entries in C generally becoming small as the optimization iteration progresses. That is, the regularization is successively reduced as the iterates get closer to the minimum. For the Stokes problem, the entries of C are generally small since they scale track solutions to the (perturbed) optimality conditions

$$\nabla f(x) = B^T(x)y \quad \text{and} \quad Yc(x) = \mu e, \tag{5}$$

where y are Lagrange multipliers (dual variables), e is the vector of ones,

$$B(x) = \nabla c(x) \quad \text{and} \quad Y = \operatorname{diag}\{y_1, y_2, \ldots, y_m\},$$

as the positive scalar parameter μ is decreased to zero. The Newton correction $(\Delta x, \Delta y)$ to the solution estimate (x, y) of (5) satisfy the equation [3]:

$$\begin{pmatrix} A(x,y) & -B^T(x) \\ YB(x) & C(x) \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} -\nabla f(x) + B^T(x)y \\ -Yc(x) + \mu e \end{pmatrix},$$

where

$$A(x,y) = \nabla_{xx} f(x) -$$

2 Preconditioning A_c by P

Suppose that we precondition A_c by P; where P is defined in (3). The decision to investigate this form of preconditioner is motivated in Section 1. We shall use the following assumptions in our theorems:

A1 $B \in \mathbb{R}^{m \times n}$ $(m < n)$ has full rank,

A2 C has rank $p > 0$ and is factored as EDE^T; where $E \in \mathbb{R}^{m \times p}$ and has orthonormal columns, and $D \in \mathbb{R}^{p \times p}$ is non-singular,

A3 If $p < m$; then $F \in \mathbb{R}^{m \times (m-p)}$ is such that its columns form a basis for the nullspace of C and the columns of $N \in \mathbb{R}^{n \times (n-m+p)}$ form a basis of the nullspace of $F^T B$;

A4 If $p = m$; then $N = I \in \mathbb{R}^{n \times n}$.

Theorem 2.1. *Assume that*

where $x \neq 0$ satisfies $Ax = Gx.$ There is no guarantee that such an eigenvector will exist, and therefore no guarantee that there are any unit eigenvalues.

If $\lambda \neq 1,$ then Equation (8) and the non-singularity of C gives

$$y = (1 - \lambda)C^{-1}Bx; \quad x \neq 0.$$

By substituting this into (7) and rearranging we obtain the quadratic eigenvalue problem

$$\left(\lambda^2 B^T C^{-1} B - \lambda\left(G + 2B^T C^{-1} B\right) + A + B^T C^{-1} B\right) x = 0. \tag{9}$$

The non-unit eigenvalues of (6) are therefore defined by the finite (non-unit) eigenvalues of (9). Note that since $B^T C^{-1} B$ has rank $m,$ (9) has $2n - (n - m) = n + m$ finite eigenvalues, but at most n linearly independent eigenvectors [22, Section 3.1]. Hence, $P^{-1}A_c$ has at most n linearly independent eigenvectors associated with the non-unit eigenvalues when $p = m.$

Now, assumption A2 implies that

$$C^{-1} = ED^{-1}E^T,$$

and, hence, letting $w_{n1} = x$ we complete our proof for the case $p = m.$

<u>Case $0 < p < m$</u> Any $y \in \mathbb{R}^m$ can be written as $y = Ey_e + Fy_f.$ Substituting this into (6) and premultiplying the resulting generalized eigenvalue problem by

$$\begin{bmatrix} I & 0 \\ 0 & E^T \\ 0 & F^T \end{bmatrix},$$

we obtain

$$\left[\begin{array}{cc|c} A & B^T E & B^T F \\ E^T B & -D & 0 \\ \hline F^T B & 0 & 0 \end{array}\right] \begin{bmatrix} x \\ y_e \\ y_f \end{bmatrix} = \lambda \left[\begin{array}{cc|c} G & B^T E & B^T F \\ E^T B & 0 & 0 \\ \hline F^T B & 0 & 0 \end{array}\right] \begin{bmatrix} x \\ y_e \\ y_f \end{bmatrix}. \tag{10}$$

Noting that the (3,3) block has dimension $(m - p) \times (m - p)$ and is a zero matrix in both coefficient matrices, we can apply Theorem 2.1 from [16] to obtain:

- $P^{-1}A_c$ has an eigenvalue at 1 with multiplicity $2(m - p);$
- the remaining $n - m + 2p$ eigenvalues are defined by the generalized eigenvalue problem

$$\overline{N}^T \begin{bmatrix} A & B^T E \\ E^T B & -D \end{bmatrix} \overline{N} w_n = \lambda \overline{N}^T \begin{bmatrix} G & B^T E \\ E^T B & 0 \end{bmatrix} \overline{N} w_n, \tag{11}$$

where $\overline{N}$ is an $(n+p) \times (n - m + 2p)$ basis for the nullspace of $\begin{bmatrix} F^T B & 0 \end{bmatrix}.$

One choice for $\overline{N}$ is

$$\overline{N} = \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix}.$$

Substituting this into (11) we obtain the generalized eigenvalue problem

$$\begin{bmatrix} N^T A N & N^T B^T E \\ E^T B N & -D \end{bmatrix} \begin{bmatrix} w_{n1} \\ w_{n2} \end{bmatrix} = \lambda \begin{bmatrix} N^T G N & N^T B^T E \\ E^T B N & 0 \end{bmatrix} \begin{bmatrix} w_{n1} \\ w_{n2} \end{bmatrix}. \tag{12}$$

This generalized eigenvalue problem resembles that of (6) in the first case considered in this proof. Therefore, the non-unit eigenvalues of $P^{-1}A_c$ are equal to the finite (and non-unit) eigenvalues of the quadratic eigenvalue problem

$$\begin{aligned} 0 \;=\; & \lambda^2 N^T B^T E D^{-1} E^T B N w_{n1} - \lambda N^T (G + 2B^T E D^{-1} E^T B) N w_{n1} \\ & + N^T (A + B^T E D^{-1} E^T B) N w_{n1}. \end{aligned} \tag{13}$$

Since $N^T B^T E D^{-1} E^T B N$ has a nullspace of dimension $n - m$; this quadratic eigenvalue problem has $2(n - m + p) - (n - m) = n - m + 2p$ finite eigenvalues [22].

□

The following numerical examples illustrate how the rank of C dictates a lower bound on the number of unit eigenvalues. In particular, Example 2.2 demonstrates that there is no guarantee that the preconditioned matrix has unit eigenvalues when C is nonsingular.

Example 2.2 (C **nonsingular**).

Consider the matrices

$$A_c = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \quad P = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

so that $m = p = 1$ and $n = 2$. The preconditioned matrix $P^{-1}A_c$ has eigenvalues at $\frac{1}{2}$, $2 - \sqrt{2}$ and $2 + \sqrt{2}$. The corresponding eigenvectors are $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$, $\begin{bmatrix} 1 & 0 & (\sqrt{2} - 1) \end{bmatrix}^T$ and $\begin{bmatrix} 1 & 0 & -(\sqrt{2} + 1) \end{bmatrix}^T$ respectively. The preconditioned system $P^{-1}A_c$ has all non-unit eigenvalues, but this does not go against Theorem 2.1 because $m - p = 0$. With our choices of A_c and P, and setting $D = [1]$ and $E = [1]$ ($C = EDE^T$), the quadratic eigenvalue problem (13) is

$$\left(\lambda^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0.$$

This quadratic eigenvalue problem has three finite eigenvalues which are $\lambda = \frac{1}{2}$, $\lambda = 2 - \sqrt{2}$ and $\lambda = 2 + \sqrt{2}$.

Example 2.3 (C **semidefinite**).

Consider the matrices

$$A_c = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \quad P = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

so that $m = 2$, $n = 2$ and $p = 1$. The preconditioned matrix $P^{-1}A_c$ has two unit eigenvalues and a further two at $\lambda = 2 - \sqrt{2}$ and $\lambda = 2 + \sqrt{2}$. There is just one linearly independent eigenvector associated with the unit eigenvector; specifically this is $\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T$. For the non-unit eigenvalues, the eigenvectors are $\begin{bmatrix} 0 & 1 & 0 & (\sqrt{2} - 1) \end{bmatrix}^T$ and $\begin{bmatrix} 0 & 1 & 0 & -(\sqrt{2} + 1) \end{bmatrix}^T$ respectively.

Since $2(m - p) = 2$, we correctly expected there to be at least two unit eigenvalues, Theorem 2.1. The remaining eigenvalues will be defined by the quadratic eigenvalue problem (13):

$$\left(\lambda^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0, \quad u_2 \neq 0,$$

where $D = [1]$ and $E = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ are used as factors of C. This quadratic eigenvalue problem has three finite eigenvalues $\lambda = 2$

Proof.

Rearranging we find that we require

$$w_{n1}^T N^T G N w_{n1} > w_{n1}^T N^T A N w_{n1}$$

for all $w_{n1} \neq 0.$ Thus we need only scale any positive definite G such that $\frac{w_{n1}^T N^T G N w_{n1}}{w_{n1}^T N^T N w_{n1}} > \|A\|_2^2$ for all $N w_{n1} \neq 0$ to guarantee that (16) is positive for all w_{n1} such that $\|w_{n1}\|_{N^T A N + \bar{D}} = 1.$ For example, we could choose $G = \alpha I,$ where $\alpha > \|A\|_2^2.$

Using the above in conjunction with Theorem 2.1 we obtain the following result:

Theorem 2.5. *Suppose that A1–A4 hold and $\bar{D}$ is as defined in (15). Further, assume that $A + \bar{D}$ and $G + 2\bar{D}$ are symmetric positive definite, $\bar{D}$ is symmetric positive semidefinite and*

$$\min \left\{ (z^T G z)^2 + 4(z^T \bar{D} z)(z^T G z + z^T \bar{D} z - 1) : \|z\|_{A+\bar{D}} = 1 \right\} > 0, \qquad (17)$$

then all the eigenvalues of $P^{-1}A_c$ are real and positive. (Condition (17) is guaranteed to hold if $G = \alpha I,$ where $\alpha > \|A\|_2^2.$) The matrix $P^{-1}A_c$ also has $m - p + i + j$ linearly independent eigenvectors. There are

1. *$m - p$ eigenvectors of the form $\begin{bmatrix} 0^T & y_f^T \end{bmatrix}^T$ that correspond to the case $\lambda = 1;$*
2. *i $(0 \leq i \leq n)$ eigenvectors of the form $\begin{bmatrix} w^T & 0^T & y_f^T \end{bmatrix}^T$ arising from $Aw = \sigma G w$ for which the i vectors w are linearly independent, $\sigma = 1;$ and $\lambda = 1;$ and*
3. *j $(0 \leq j \leq n - m + 2p)$ eigenvectors of the form $\begin{bmatrix} 0^T & w_{n1}^T & w_{n2}^T & y_f^T \end{bmatrix}^T$ corresponding to the eigenvalues of $P^{-1}A_c$ not equal to 1, where the components w_{n1} arise from the quadratic eigenvalue problem*

$j \cdot$

Theorem 2.1 also shows that the eigenvectors corresponding to $\lambda \neq 1$ take the form $\begin{bmatrix} x^T & y^T \end{bmatrix}^T$; where x corresponds to the quadratic eigenvalue problem (9) and $y = (1-\lambda)C^{-1}Bx = (1-\lambda)D^{-1}EBNx$ (since we can set $D = C$ and $E = I$). Clearly, there are at most $n+m$ such eigenvectors. By our assumptions, all of the vectors x defined by the quadratic eigenvalue problem (9) are linearly independent. Also, if x is associated with two eigenvalues, then these eigenvalues must be distinct [22]. By setting $w_{n1} = x$ and $w_{n2} = y$ we obtain j ($0 \le j \le n+m$) eigenvectors of the form given in statement 3 of the proof.

It remains for us to prove that the $i+j$ eigenvectors defined above are linearly independent. Hence, we need to show that

$$\begin{bmatrix} x_1^{(1)} & \cdots & x_i^{(1)} \\ 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} a_1^{(1)} \\ \vdots \\ a_i^{(1)} \end{bmatrix} + \begin{bmatrix} x_1^{(2)} & \cdots & x_j^{(2)} \\ y_1^{(2)} & \cdots & y_j^{(2)} \end{bmatrix} \begin{bmatrix} a_1^{(2)} \\ \vdots \\ a_j^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{18}$$

implies that the vectors $a^{(1)}$ and $a^{(2)}$ are zero vectors. Multiplying (18) by $P^{-1}A_c$; and recalling that in the previous equation the first matrix arises from $\lambda_l = 1$ ($l = 1, \cdots, i$) and the second matrix from $\lambda_l \neq 1$ ($l = 1, \cdots, j$) gives

$$\begin{bmatrix} x_1^{(1)} & \cdots & x_i^{(1)} \\ 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} a_1^{(1)} \\ \vdots \\ a_i^{(1)} \end{bmatrix} + \begin{bmatrix} x_1^{(2)} & \cdots & x_j^{(2)} \\ y_1^{(2)} & \cdots & y_j^{(2)} \end{bmatrix} \begin{bmatrix} \lambda_1^{(2)} a_1^{(2)} \\ \vdots \\ \lambda_j^{(2)} a_j^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{19}$$

Subtracting (18) from (19) we obtain

$$\begin{bmatrix} x_1^{(2)} & \cdots & x_j^{(2)} \\ y_1^{(2)} & \cdots & y_j^{(2)} \end{bmatrix} \begin{bmatrix} (\lambda_1^{(2)} - 1)a_1^{(2)} \\ \vdots \\ (\lambda_j^{(2)} - 1)a_j^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{20}$$

Some of the eigenvectors x defined by the quadratic eigenvalue problem (9) and

$$a_l^{(2)} = -a_{l+k}^{(2)} \frac{1 - \lambda_{l+k}^{(2)}}{1 - \lambda_l^{(2)}}; \quad l = 1, \ldots, k.$$

Now $y_l^{(2)} = (1 - \lambda_l^{(2)}) C^{-1} B x_l^{(2)}$ for $l = 1, \ldots, 2k.$ Hence, we require

$$(\lambda_1^{(2)} - 1)^2 a_l^{(2)} C^{-1} B x_l^{(2)} + (\lambda_1^{(2)} - 1)^2 a_{l+k}^{(2)} C^{-1} B x_l^{(2)} = 0; \quad l = 1, \ldots, k.$$

Substituting in $a_l^{(2)} = -a_{l+k}^{(2)} \frac{1 - \lambda_{l+k}^{(2)}}{1 - \lambda_l^{(2)}}$ and rearranging gives $(\lambda_l^{(2)} - 1) a_l^{(2)} = (\lambda_{l+k}^{(2)} - 1) a_{l+k}^{(2)}$ for $l = 1, \ldots, k.$

From (25), it may be deduced that either $\lambda = 1$ or $w_m = 0.$ In the former case, (23) and (24) may be simplified to

$$Q^T \begin{bmatrix} A & B^TE \\ E^TB & -D \end{bmatrix} Qw = Q^T \begin{bmatrix} G & B^TE \\ E^TB & 0 \end{bmatrix} Qw, \qquad (26)$$

where $Q = \begin{bmatrix} \bar{M} & \bar{N} \end{bmatrix}$ and $w = \begin{bmatrix} w_m^T & w_n^T \end{bmatrix}.$ Since Q is orthogonal, the general eigenvalue problem (26) is equivalent to considering

$$\begin{bmatrix} A & B^TE \\ E^TB & -D \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \sigma \begin{bmatrix} G & B^TE \\ E^TB & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \qquad (27)$$

where $\begin{bmatrix} w_1^T & w_2^T \end{bmatrix}^T \neq 0$ if and only if $\sigma = 1,$ and $w_1 \in \mathbb{R}^n,$ $w_2 \in \mathbb{R}^p.$ As in the first case of this proof, nonsingularity of D and $\sigma = 1$ implies that $w_2 = 0.$ There are $m - p$ linearly independent eigenvectors $\begin{bmatrix} 0^T & 0^T & u_f^T \end{bmatrix}^T$ corresponding to $w_1 = 0,$ and a further i $(0 \leq i \leq n)$ linearly independent eigenvectors corresponding to $w_1 \neq 0$ and $\sigma = 1.$

Now suppose that $\lambda \neq 1,$ in which case $w_m = 0.$ Equations (23) and (24) yield

$$\bar{N}^T \begin{bmatrix} A & B^TE \\ E^TB & -D \end{bmatrix} \bar{N} w_n = \lambda \bar{N}^T \begin{bmatrix} G & B^TE \\ E^TB & 0 \end{bmatrix} \bar{N} w_n, \qquad (28)$$

$$\bar{M}^T \begin{bmatrix} A & B^TE \\ E^TB & -D \end{bmatrix} \left(\bar{N} w_n + \bar{R} y_f \right) = \lambda \bar{M}^T \begin{bmatrix} G & B^TE \\ E^TB & 0 \end{bmatrix} \left(\bar{N} w_n + \bar{R} y_f \right). \qquad (29)$$

The generalized eigenvalue problem (29) defines $n - m + 2p$ eigenvalues, where j $(0 \leq j \leq n - m)$ of these are not equal to 1 and for which two cases have to be distinguished. If $w_n = 0,$ then (28) and $\lambda \neq 1$ imply that $y_f = 0.$ In this case no extra eigenvalues arise. Suppose that $w_n \neq 0,$ then, from the proof of Theorem 2.1, the eigenvalues are equivalently defined by (13) and

$$w_n = \begin{bmatrix} w_{n1} \\ (1 - \lambda) D^{-1} E^T B N w_{n1} \end{bmatrix}.$$

Hence, the j $(0 \leq j \leq n - m + 2l)$ eigenvectors corresponding to the non-unit eigenvalues of $P^{-1}A_c$ take the form $\begin{bmatrix} 0^T & w_{n1}^T & w_{n2}^T & y_f^T \end{bmatrix}^T.$

Proof of the linear independence of these eigenvectors follows similarly to the case of $p = m.$ □

Observing that the coefficient matrices in (10) are of the form of those considered by Gould, Hribar and Nocedal [12], we could apply a projected preconditioned conjugate gradient method to solve (1) if all the eigenvalues of $P^{-1}A_c$ are real and positive and we have a decomposition of C as in A2. Theorem 2.5 therefore gives conditions which allow us to use such a method. Dollar gives a variant of this method in which no decomposition of C is required, see [6, Section 5.5]. The derivation of such a method bears close resemblance to that

3 Convergence

In the context of this paper, the convergence of an iterative method under preconditioning is not only influenced by the spectral properties of the coefficient matrix, but also by the relationship between m, n and $p.$ We can determine an upper bound on the number of iterations of an appropriate Krylov subspace method by considering minimum polynomials of the coefficient matrix.

Definition 3.1. *Let $A \in \mathbb{R}^{(n+m)\times(n+m)}.$ The monic polynomial f of minimum degree such that $f(A) = 0$ is called the minimum polynomial of $A.$*

Krylov subspace theory states that iteration with any method with an optimality property, e.g. GMRES, will terminate when the degree of the minimum polynomial is attained, [21]. In particular, the degree of the minimum polynomial is equal to the dimension of the corresponding Krylov subspace (for general b), [20, Proposition 6.1].

Theorem 3.2. *Suppose that the assumptions of Theorem 2.5 hold. The dimension of the Krylov subspace $\mathcal{K}(\mathcal{P}^{-1}\mathcal{A}_c, b)$ is at most $\min\{n-m+2p+2, n+m\}.$*

Proof. Suppose that $0 < p < m.$ As in the proof to Theorem 2.1, the generalized eigenvalue problem can be written as

$$\left[\begin{array}{cc|c} A & B^TE & B^TF \\ E^TB & -D & 0 \\ \hline F^TB & 0 & 0 \end{array}\right]\begin{bmatrix} x \\ y_e \\ y_f \end{bmatrix} = \lambda \left[\begin{array}{cc|c} G & B^TE & B^TF \\ E^TB & 0 & 0 \\ \hline F^TB & 0 & 0 \end{array}\right]\begin{bmatrix} x \\ y_e \\ y_f \end{bmatrix}. \quad (30)$$

Hence, the preconditioned matrix $\mathcal{P}^{-1}\mathcal{A}_c$ can be written as

$$\mathcal{P}^{-1}\mathcal{A}_c = \begin{bmatrix} \Theta_1 & 0 \\ \Theta_2 & I \end{bmatrix}, \quad (31)$$

where the precise forms of $\Theta_1 \in \mathbb{R}^{(n+p)\times(n+p)}$ and $\Theta_2 \in \mathbb{R}^{(m-p)\times(n+p)}$ are irrelevant.

From the earlier eigenvalue derivation, it is evident that the characteristic polynomial of the preconditioned linear system (31) is

$$\left(\mathcal{P}^{-1}\mathcal{A}_c - I\right)^{2(m-p)} \prod_{i=1}^{n-m+2p} \left(\mathcal{P}^{-1}\mathcal{A}_c - \lambda_i I\right).$$

In order to prove the upper bound on the Krylov subspace dimension, we need to show that the order of the minimum polynomial is less than or equal to $\min\{n - m+2p+2, n+m\}.$ Expanding the polynomial $\left(\mathcal{P}^{-1}\mathcal{A}_c - I\right)\prod_{i=1}^{n-m+2p}\left(\mathcal{P}^{-1}\mathcal{A}_c - \lambda_i I\right)$ of degree $n - m + 2p + 1,$ we obtain

$$\begin{bmatrix} (\Theta_1 - I)\prod_{i=1}^{n-m+2p}(\Theta_1 - \lambda_i I) & 0 \\ \Theta_2 \prod_{i=1}^{n-m+2p}(\Theta_1 - \lambda_i I) & 0 \end{bmatrix}.$$

Since the assumptions of Theorem 2.5 hold, Γ_1 has a full set of linearly independent eigenvectors and is diagonalizable. Hence, $(\Gamma_1 - I)\prod_{i=1}^{n-m+2p}(\Gamma_1 - \lambda_i I) = 0.$ We therefore obtain

$$\left(P^{-1}A_c - I\right)\prod_{i=1}^{n-m+2p}\left(P^{-1}A_c - \lambda_i I\right) = \begin{bmatrix} 0 & 0 \\ \Gamma_2 \prod_{i=1}^{n-m+2p}(\Gamma_1 - \lambda_i I) & 0 \end{bmatrix}. \quad (32)$$

If $\Gamma_2 \prod_{i=1}^{n-m+2p}(\Gamma_1 - \lambda_i I$ $: :=1$ $(\Gamma_1 - \lambda_i I) =$ TD[(0)]TJ/F139.96Tf4.980TD[(:)]TJ/F;9.96Tf5.160TD[(P $I:$

5]TJ/F15 9.96 Tf 6.92 -14.29D[(5]TJ/F;0)]67(c2 9.96 Tf 6.13 8.4974+[(¢)]TJ/F4 9.96 Tf 7.33 -867.33D[(P)]TJ/F(

If $\frac{\delta-\epsilon}{\gamma} \geq 0$, then

$$\sqrt{\left(\frac{\delta}{2\gamma}\right)^2 + \frac{\delta-\epsilon}{\gamma}} \leq \sqrt{2\max\left\{\left(\frac{\delta}{2\gamma}\right)^2, \frac{\delta-\epsilon}{\gamma}\right\}}$$
$$= \sqrt{2}\max\left\{\frac{\delta}{2\gamma}, \sqrt{\frac{\delta-\epsilon}{\gamma}}\right\}.$$

If $\frac{\delta-\epsilon}{\gamma} \leq 0$, then the assumption $\delta^2 + 4\gamma(\delta-\epsilon) \geq 0$ implies that

$$\left(\frac{\delta}{2\gamma}\right)^2 \geq \frac{\epsilon-\delta}{\gamma} \geq 0.$$

Hence,

$$\sqrt{\left(\frac{\delta}{2\gamma}\right)^2 + \frac{\delta-\epsilon}{\gamma}} \leq \frac{\delta}{2\gamma}$$
$$< \sqrt{2}\max\left\{\frac{\delta}{2\gamma}, \sqrt{\frac{\epsilon-\delta}{\gamma}}\right\}.$$

$\square$

Remark 3.4. *The important point to notice is that if $\gamma \gg \delta$ and $\gamma \gg \epsilon$, then $\lambda \approx 1$ in Theorem 3.3.*

Theorem 3.5. *Assume that the assumptions of Theorem 2.5 hold, then all the eigenvalues of $P^{-1}A_c$ are real and positive, and $2(m-p)$ of them are guaranteed to be equal to 1. In addition, the eigenvalues λ of (13) subject to $E^TBNu \neq 0$, will also satisfy*

$$|\lambda - 1| = O(\max\{\|C\|, \|G-A\|\sqrt{\|C\|}\})$$

for small values of $\|C\|$.

Proof. That the eigenvalues of $P^{-1}A_c$ are real and positive follows directly from Theorem 2.5.

Suppose that $C = EDE^T$ is a reduced singular value decomposition of C, where the columns of $E \in \mathbb{R}^{m\times p}$ are orthogonal and $D \in \mathbb{R}^{p\times p}$ is diagonal with entrieduced singula9eThat the 9eThat the $-$

Premultiplying the quadratic eigenvalue problem (13) by u^T gives

$$0 = \lambda^2 u^T \hat{D} u - \lambda (u^T N^T G N u + 2 u^T \hat{D} u) + (u^T N^T A N u + u^T \hat{D} u). \quad (33)$$

Assume that $v = E^T B N u$ and $\|v\| = 1$, where u is an eigenvector of the above quadratic eigenvalue problem, then

$$\begin{aligned} u^T \hat{D} u &= v^T D^{-1} v \\ &= \frac{v_1^2}{d_1} + \frac{v_2^2}{d_2} + \dots + \frac{v_m^2}{d_m} \\ &\geq \frac{v^T v}{d_1} \\ &= \frac{1}{\|C\|}. \end{aligned}$$

Hence,

$$\frac{1}{u^T \hat{D} u} \leq \|C\|.$$

Let $\gamma = u^T \hat{D} u$, $\delta = u^T N^T G N u$ and $\varepsilon = u^T N^T A N u$, then (33) becomes

$$\lambda^2 \gamma - \lambda(\delta + 2\gamma) + \varepsilon + \gamma = 0.$$

From Theorem 3.3, λ must satisfy

$$\lambda = 1 + \frac{\delta}{2\gamma} \pm \xi, \quad \xi \leq \sqrt{2} \max \left\{ \frac{\delta}{2\gamma}, \sqrt{\frac{|\delta - \varepsilon|}{\gamma}} \right\}.$$

Now $\delta \leq c \left\| N^T G N \right\|$, $\varepsilon \leq c \left\| N^T A N \right\|$, where c is an upper bound on $\|u\|$ and u are eigenvectors of (13) subject to $\left\| E^T B N u \right\| = 1$. Hence, the eigenvalues of (13) subject to $E^T B N u \neq 0$ satisfy

$$|\lambda - 1| = O(\max\{\|C\|, \|G - A\| \sqrt{\|C\|}\})$$

for small values of $\|C\|$. □

The results of this theorem are not very surprising, but basic eigenvalue perturbation theorems such as Theorem 7.7.2 in [10] in conjunction with Theorem 2.3 of [16] are weaker than what we have established. Specifically, the structure of our coefficient matrix and preconditioner means that we are still guaranteed to have $2(m - p)$ unit eigenvalues, whereas the more general eigenvalue perturbation theorems would only imply that these eigenvalues will be close to 1.

Example 3.6 (C with small entries).

Suppose that A_c and P are as in Example 2.2, but $C = [10^{-a}]$ for some positive real number $a.$ Setting $D = [10^{-a}]$ and $E = [1]$ ($C = EDE^T$), the quadratic eigenvalue problem (13) is

$$\left(\lambda^2 \begin{bmatrix} 10^a & 0 \\ 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 2 + 2 \times 10^a & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 + 10^a & 0 \\ 0 & 1 \end{bmatrix}\right) \begin{bmatrix} x_y \\ x_z \end{bmatrix} = 0.$$

This quadratic eigenvalue problem has three finite eigenvalues: $\lambda = \frac{1}{2},$

$$\lambda = 1 + 10^{-a} \pm 10^{-a}\sqrt{1 + 10^a}.$$

For large values of $a,$ $\lambda \approx 1 + 10^{-a} \pm 10^{-\frac{a}{2}}$; the eigenvalues will be close to 1.

This clustering of part of the spectrum of $P^{-1}A_c$ will often translate into a speeding up of the convergence of a selected Krylov subspace method, [1, Section 1.3].

3.2 Numerical Examples

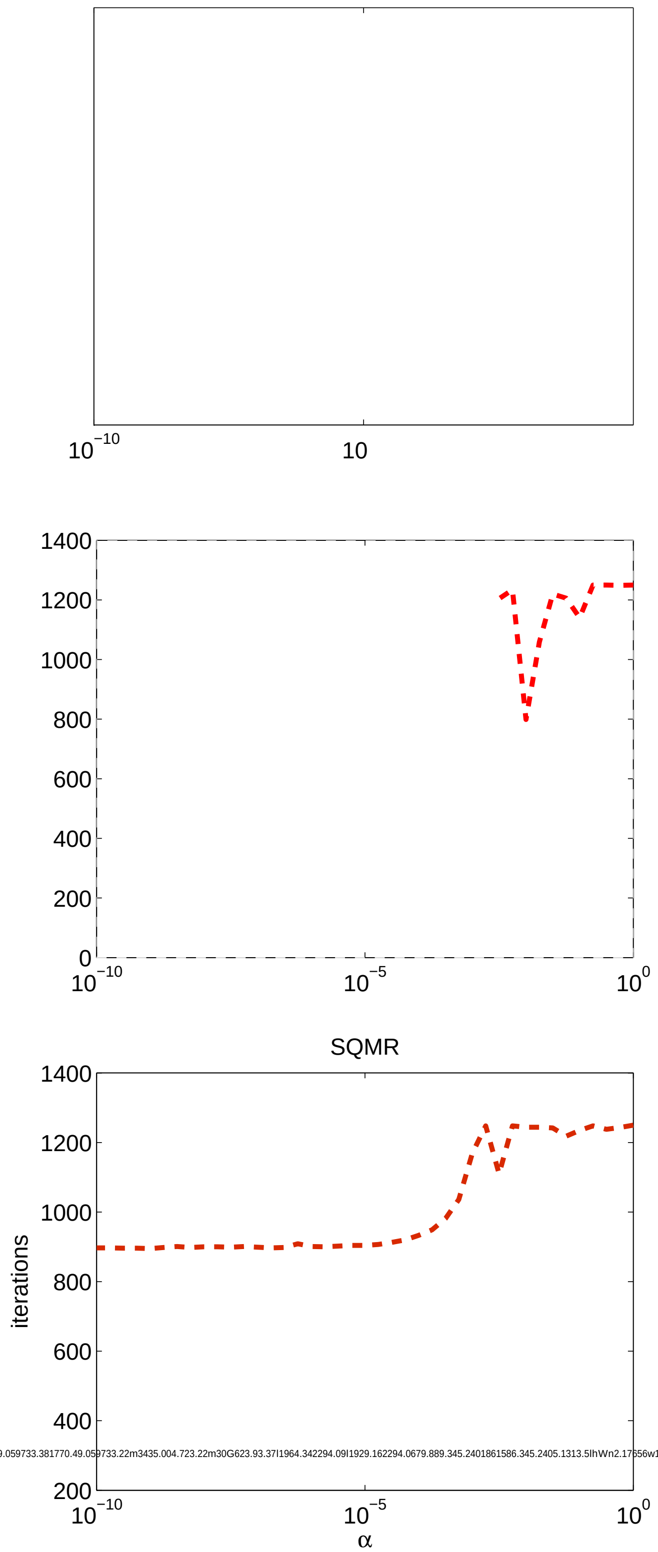
We will carry out several numerical tests to verify that, in practice, our theoretical results translate to a speeding up in the convergence of a selected Krylov subspace method as the entries of C converge towards 0.

Example 3.7.

The CUTEr test set [13] provides a set of quadratic programming problems. We shall use the problem CVXQP2_M in the following two examples. This problem has $n = 1000$ and $m = 250.$ "Barrier" penalty terms (in this case $\alpha,$ where α is defined below) are added to the diagonal of A to simulate systems that might arise during an iteration of an interior-point method for such problems. We shall set $G = diag(A)$ (ignoring the additional penalty terms), and $C = \alpha I;$ where α is a positive, real parameter that we will change.

All tests were performed on a dual Intel Xeon 3.20GHz machine with m
CC

SQMR method doesn't have an optimality property as was assumed in Sec-



a)5.08131313.5504.723.38177019.059733.381770.49.059733.22m3435.004.723.22m30G623.93.37I1964.342294.09I1929.162294.0679.889.345.2401861586.345.2405.1313.5lhWn2.17656w1i1i1G623.9411313.911411.131097.71reS0G623.9(1400)TiETO623.9412411.63m2035.082411.63I8177.835G

Figure 1: Comparison of number of iterations required when either (a) P or (b) P_c are used as preconditioners for $C = ®I$ with GMRES, PPCG and SQMR on the CVXQP2_M problem.

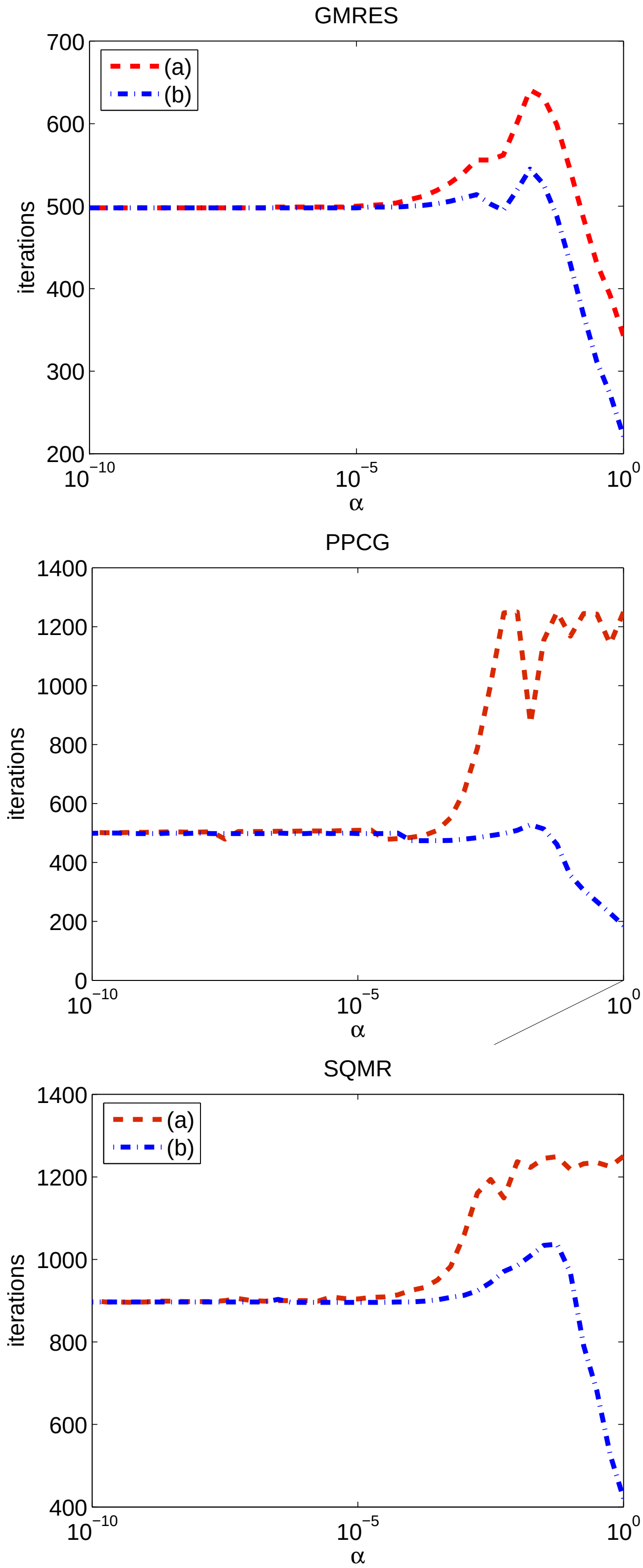


Figure 2: Comparison of number of iterations required when either (a) P or (b) P_C are used as preconditioners for C = ®I with GMRES, PPCG and SQMR on the CVXQP2_M problem.

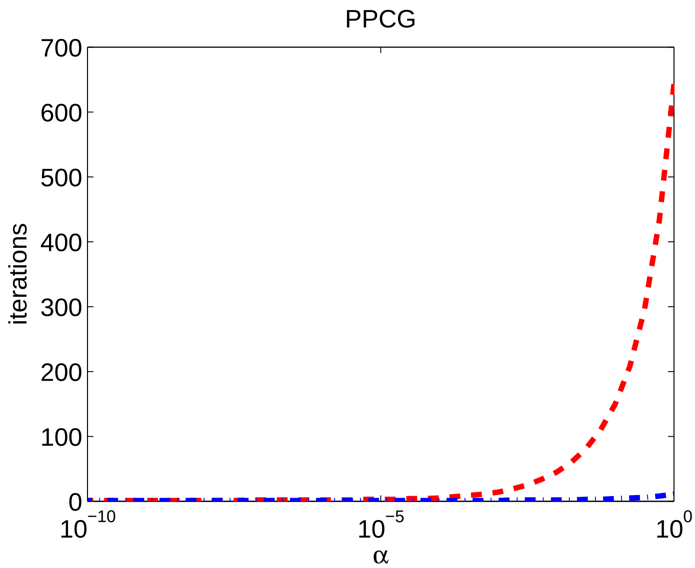


Figure 3: Number of PPCG iterations when either (a) P or (b) P_c are used as preconditioners for C = ®I on the AUG2DQP problem.

Figure 4: Number of PPCG iterations when either (a) P or (b) P_c are used as preconditioners for C = ® £ $diag$(0; : : : ; 0; 1; : : : ; 1); where r$ankC$ = bm=2c; on the AUG2DQP problem.

These examples suggest that during pre-asymptotic iterations of an interior point method for a nonlinear programming problem, we may need to use a preconditioner of the form P_c; but as the method proceeds there will be a point at which we will be able to swap to using a preconditioner of the form P. From this point onwards, we'll be able to use the same preconditioner during each iterative solve of the resulting sequence of saddle-point problems.

4 Conclusion and further research

In this paper, we have investigated a class of preconditioners for indefinite linear systems that incorporate the (1,2) and (2,1) blocks of the original matrix. These blocks are often associated with constraints. We have shown that if C has rank $p > 0$; then the preconditioned system has at least $2(m - p)$ unit eigenvalues, regardless of the structure of G. In addition, we have shown that if the entries of C are very small, then we will expect an additional $2p$ eigenvalues to be clustered around 1 and, hence, for the number of iterations required by our chosen Krylov subspace method to be dramatically reduced. These later results are of particular relevance to interior point methods for optimization.

The practical implications of the analysis of this paper in the context of solving nonlinear programming problems will be the subject of a follow-up paper. We will investigate the point at which the user should switch from using a preconditioner of the form P_c to that of P during an interior point method, and how the sub-matrix G in the preconditioner should be chosen.

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