1 Introduction

In recent years, bed-load sediment transport in channels and rivers has become a major problem. The understanding as to how sediment interacts in certain environments is crucial for both the environment and businesses. For example, if a dam is constructed such that sediment is deposited at the entrance of the dam, a build up of sediment could occur resulting in the sediment needing to be drained frequently. The walls of the dam may get severely damaged if there is a massive build up of sediment next to the walls. In the environment, the understanding of how sediment interacts is essential when chemicals are deposited into rivers. e need to be able to make sure that these chemicals are being dispersed safely and how they interact with the river.

Throughout this report, we will be discussing how we can accurately numerically approximate the equations governing bed-load sediment transport in channels and rivers. These comprise the equation for conservation

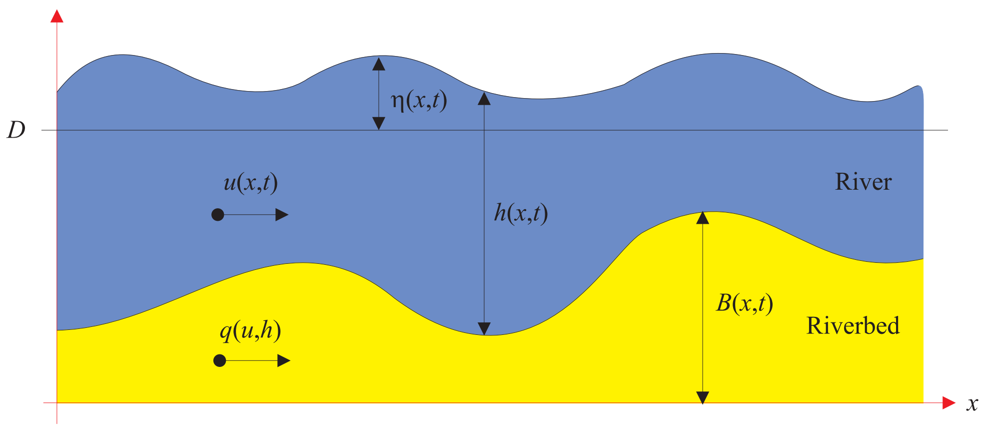


Figure 1.1: The variables $h\ x,t)$, $\ x,t)$, $\eta\ x,t)$, $\ ,h)$ and $B\ x,t)$.

The value of A usually depends on the depth of the river andAs]R [flJJJM BecUSff:SAfl]5] fff

2 Different Formulations

There are numerous ways we can re-write the system 1.1), 1.2) and 1.3) but we require a formulation of the system that can be used to obtain an accurate numerical approximation. All of the formulations discussed in this section will be derived using Sediment Transport Flux 1.5), thus we also require that

$$h\ x,t) > 0, \qquad x,t) \geq 0 \quad \text{and} \quad -\infty < B\ x,t) < \infty.$$

All systems discussed in this section can be written in the form

$$\frac{\partial \mathbf{w}}{\partial t} + \frac{\partial \mathbf{F}\ \mathbf{w})}{\partial x} = \mathbf{R}, \tag{2.1}$$

where $\mathbf{R}$ is the source term and $\mathbf{F}$ is the flux function, which is called a non-homogeneous conservation law with source term present and whose Jacobian matrix is $\mathbf{J} = \frac{\partial \mathbf{F}}{\partial \mathbf{w}}$. Some of the numerical schemes discussed in Section 3 require the eigenvalues and eigenvectors of the Jacobian matrix, and so we also derive these here.

2.1 Formulation A

One approach we could use is to re-write 1.1) and 1.2) in system form,

$$\begin{bmatrix} h \\ h \end{bmatrix}_t + \begin{bmatrix} h \\ h\ \ ^2 + \frac{1}{2}gh^2 \end{bmatrix}_x = \begin{bmatrix} 0 \\ -ghB_x \end{bmatrix}, \tag{2.2}$$

which is called the Shallow ater Equations and is written in conservative variable form. e can then

2. **Formulation A-CV**: e converge the Shallow ater Equations to a steady state solution and then we update the riverbed. The overall time step of this formulation is the morphological time step of the Bed-Updating Equation and the Shallow ater Equations are conv

where $c = \sqrt{gh}$ and $d = \frac{\xi}{h}Am \quad {}^{m-1}$ for $\quad \geq 0$. Notice that this Jacobian matrix is singular, which we might expect to create difficulties when implementing a numerical scheme for this formulation. The eigenvalues of the Jacobian matrix are

$$\lambda_1 = \quad - c, \quad \lambda_2 = 0 \quad \text{and} \quad \lambda_3 = \quad + c,$$

whose corresponding eigenvectors are

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ \quad - c \\ \dfrac{-cd}{\quad - c} \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{e}_3 = \begin{bmatrix} 1 \\ \quad + c \\ \dfrac{cd}{\quad + c} \end{bmatrix}.$$

matrix cannot be easily written analytically since they are the roots of the polynomial

$$P \ \lambda, \mathbf{w}) = \lambda^3 - 2 \ \ \lambda^2 + \left[\ ^2 - g \ h + d) \right] \lambda + g \ \ d = 0.$$

However, we can prove that the roots of $P \ \lambda, \mathbf{w})$ are always real for

$$h \ x, t) > 0, \qquad x, t) \geq 0 \quad \text{and} \quad -\infty < B \ x, t) < \infty$$

by using formulae for the roots of a cubic, see Spiegel & Liu[9]. For a cubic equation,

$$P \ x) = x^3 + a_1 x^2 + a_2 x + a_3 = 0,$$

if we let

$$= \frac{1}{9} \ 3a_2 - a_1^2) \quad \text{and} \quad R = \frac{1}{54} \ 9a_1 a_2 - 27a_3 - 2a_1^3),$$

then the *discriminant* is $D = \ \ ^3 + R^2$ and if

1. $D > 0$ then one root is real and two are complex;
2. $D = 0$ then all roots are real and two are equal;
3. $D < 0$ then all roots are real and unequal.

If $D < 0$ then the roots of $P \ x)$ are

$$x_1 = 2\sqrt{- \ \ } \cos \ \frac{1}{3}\theta) - \frac{1}{3}a_1, \qquad 2.6a)$$

$$x_2 = 2\sqrt{- \ \ } \cos \ \frac{1}{3} \ \theta + 2\pi)) - \frac{1}{3}a_1 \qquad 2.6b)$$

and

$$x_3 = 2\sqrt{- \ \ } \cos \ \frac{1}{3} \ \theta + 4\pi)) - \frac{1}{3}a_1 \qquad 2.6c)$$

where $\cos\theta =$

and then combine 1.1), 2.7) and 1.3) into system form to obtain **Formulation C**,

$$\begin{bmatrix} h \\ h \\ B \end{bmatrix}_t + \begin{bmatrix} h \\ h\ ^2 + \frac{1}{2}gh^2 + ghB \\ \xi \end{bmatrix}_x = \begin{bmatrix} 0 \\ gBh_x \\ 0 \end{bmatrix}. \qquad 2.8)$$

Unfortunately, **Formulation C** has a source term present.

If we use the Sediment Transport Flux 1.5), then the Jacobian matrix of **Formulation C** is

$$\mathbf{J} = \begin{bmatrix} 0 & 1 & 0 \\ g\ h + B) - \ ^2 & 2 & gh \\ -\ d & d & 0 \end{bmatrix} \quad)$$

3 Numerical Schemes

In this section, we will discuss how we can numerically approximate the formulations discussed in the previous section. e will discuss two numerical approaches: LeVeque & Yee's MacCormack approach see LeVeque & Yee[7], Yee[13] and Hudson[6]) and an adaptation of Roe's Scheme see Roe[8], laister[3] and Hubbard & arcia-Navarro[5]). The numerical approaches discussed in this section will be based on 2.1) since all of the formulations can be written in this form. e will also try to ensure that all of the numerical approaches satisfy the Total Variational Diminishing property see Harten[4] and Sweby[11]). Flux limiter methods and slope limiter methods will be used to try to ensure that the numerical approaches satisfy the TVD property. The numerical approaches will also be adapted specifically for each formulation.

Before we can implement any numerical schemes, we must first define a mesh. In this report, we will be using a variable mesh over the finite region $x_0 \leq x \leq x_I$ and $t_0 \leq t \leq t_N$, which is illustrated in Figure 3.1. The

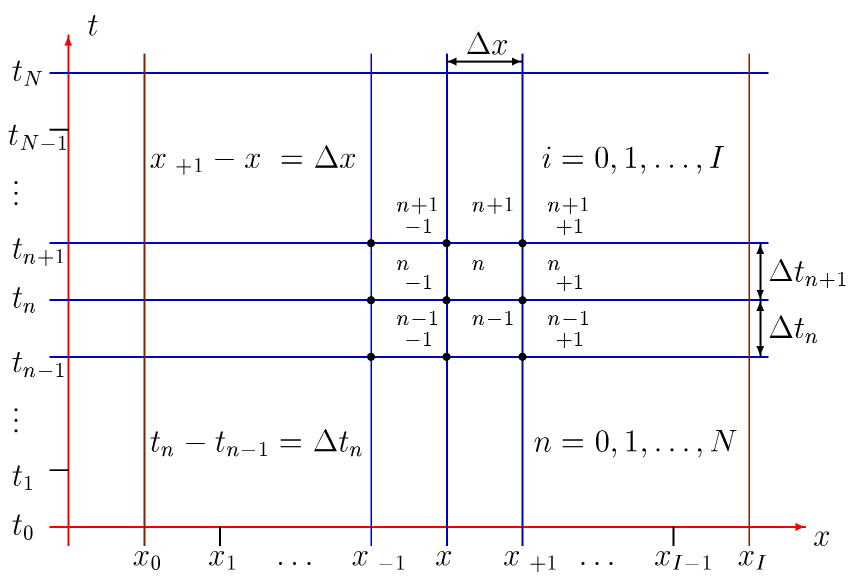


Figure 3.1: The Mesh.

points $x = x_0$ and $x = x_I$ are the spatial boundaries and we will require numerical boundary conditions at these points. The numerical solution is denoted by $\ ^n \approx \ \ i\Delta x, t_n)$, where $\Delta x = x_{\ +1} - x\ $, $t_n = t_0 + \sum_{k=1}^{n} \Delta t_k$ and $\Delta t_k = t_k - t_{k-1}$. The spatial step size, Δx, is fixed and we will use a

and

minmod

1. The

Equations to a steady state solution first and then update the riverbed. A steady state solution is obtained if

$$\frac{\partial \mathbf{w}}{\partial t} = 0.$$

Thus, in order to determine if the steady state solution has been obtained, we set a tolerance level, tol, such that

$$|\mathbf{w}^{n+1} - \mathbf{w}^{n}| \leq \mathrm{tol} \quad \text{for} \quad i = 0, 1, 2, \ldots, I.$$

The morphological time step of the Bed-Updating Equation is the overall time step of this formulation and is calculated separately from the time step of the Shallow ater Equations. The Shallow ater Equations must be converged to the steady state solution each time the riverbed is updated.

3.1.3 Form

3.1.5 Formulation C

LeVeque & Yee's MacCormack approach can be used to approximate **Formulation C** by using

$$\mathbf{w}^n = \begin{bmatrix} h^n \\ {}^n h^n \\ B^n \end{bmatrix}, \quad \mathbf{F}^n = \begin{bmatrix} h \\ h \quad {}^2 + \frac{1}{2} g h^2 + g h B \\ \xi \end{bmatrix}^n$$

and

$$\mathbf{R}^n = \left[\begin{array}{c} 0 \\ {}_{\overline{2}} \; B^n_{+1} = \end{array} \right.$$

Name of Flux-limiter	$\Phi\ \theta)$
Minmod	$\max\ 0, \min\ 1, \theta))$
Roe's Superbee	$\max\ 0, \min\ 2\theta, 1), \min\ \theta, 2))$
van Leer	$\dfrac{\lvert\theta\rvert + \theta}{1 + \lvert\theta\rvert}$
van Albada	$\dfrac{\theta^2 + \theta}{1 + \theta^2}$

Table 3.1: Some Flux-limiters

eigenvectors and wavestrengths associated with the linearised Riemann problem have been obtained, Roe's Scheme [8] can be used

$$\mathbf{w}^{n+1} = \mathbf{w}^{n} - s\ \mathbf{F}^{*}{}_{+\frac{1}{2}} - \mathbf{F}^{*}{}_{-\frac{1}{2}}) + s\mathbf{R}^{*}, \qquad 3.6)$$

where

$$\mathbf{F}^{*}{}_{+\frac{1}{2}} = \ {}^{1}$$

Hubbard & arcia-Navarro[5] and Hudson[6]). Once the values of $\tilde{\beta}_k$ have been obtained, we can approximate the source term by using

$$\mathbf{R}^* = \mathbf{R}^-_{+\frac{1}{2}} + \mathbf{R}^+_{-\frac{1}{2}}, \qquad 3.8)$$

where

$$\mathbf{R}^{\pm}_{+\frac{1}{2}} = \frac{1}{2}\sum_{k=1}^{p}[\tilde{\beta}_k \tilde{\mathbf{e}}_k \;\; 1 \pm \mathrm{sgn}\;\; \tilde{\lambda}_k) \;\; 1 - \Phi_k \;\; 1 - |\nu_k|)))]_{\;+\frac{1}{2}}.$$

The advantage of using the decomposed approach to numerically approximate the source term is that the numerical scheme satisfies the C-property of Bermudez & Vazquez[1]. For a numerical scheme to satisfy the C-property of Bermudez & Vazquez[1], the scheme must be either exact or second order accurate when applied to the quiescent flow case, i.e. $\equiv 0$ and $h \equiv$, where $\equiv D - B$. Since Roe's Scheme with source term approximation 3.8) satisfies the C-property of Bermudez & Vazquez[1], the numerical scheme will not produce any waves in a region that should remain steady see Bermudez & Vazques[1] and Hudson[6] for more details). Unfortunately, if we use the decomposed approach with the flux-limited second order version of 3.6), then spurious oscillations may occur in the numerical results when the source term is stiff see Hudson[6] and Hubbard & arcia-Navarro[5] for more details). If the spurious oscillations do overpower the numerical solution, then we can reduce them by using a smaller Courant number, but sometimes the Courant number required can be very small resulting in the numerical scheme being impractical due to long computational run times.

Roe's Scheme requires that

$$\frac{\Delta t}{\Delta x}\max\;|\lambda_k|) \leq 1$$

so that the numerical scheme will remain stable. Hence, for all of the formulations we will use

$$\Delta t = \frac{\Delta x}{2\max\;|\lambda_k^n|)}, \qquad 3.9)$$

to calculate the time step, which will ensure that the numerical scheme does not become unstable.

In the next few sections, we will discuss how we can apply 3.6) to the different formulations discussed in Section 2.

2. The

3.2.3 Formulation A- F

e can use 3.6) to approximate **Formulation A- F** by using

$$\mathbf{w}^n = \begin{bmatrix} h^n \\ {}^n h^n \\ B^n \end{bmatrix} \quad \text{and} \quad \mathbf{F}^n = \begin{bmatrix} h \\ h \quad {}^2 + \frac{1}{2} g h^2 \\ \xi A \quad {}^m \end{bmatrix}^n.$$

The eigenv

For the source term approximation, we can use either the pointwise approach,

$$\mathbf{R}^{n}_{+1}$$

If the Sediment Transport Flux 1.5) is used with $A = 1$ and m is an integer, then we can use

$$\tilde{d} = \xi \sum_{k=0}^{m-1} \quad {}_{R})^{k} \quad {}_{L})^{m-(1+k)}.$$

Also for **Formulation B**, there is no source term present so

$$\mathbf{R}^{n}_{+\frac{1}{2}} = 0 \quad \text{and} \quad \tilde{\beta}_k = 0.$$

3.2.5 Formulation C

Formulation C can also be approximated by using 3.6) with

$$\mathbf{w}^n = \begin{bmatrix} h^n \\ {}^{n}h^n \\ B^n \end{bmatrix} \quad \text{and} \quad \mathbf{F}^n = \begin{bmatrix} h \\ h \quad {}^{2} + \frac{1}{2}gh^2 + ghB \\ \xi A \quad {}^{m} \end{bmatrix}^n.$$

The Roe averaged eigenvalues are obtained by finding the roots of

$$\tilde{P} \ \tilde{\lambda}) = \tilde{\lambda}^3 - 2\tilde{\ }\tilde{\lambda}^2 + [\tilde{\ }^2 - g \ \tilde{h} + \tilde{B} + \tilde{h}\tilde{d})]\tilde{\lambda} + g\tilde{h}\tilde{\ }\tilde{d} = 0.$$

The roots of $\tilde{P} \ \tilde{\lambda})$ are determined by using the approach which was discussed in Section 2.2. Once the values of the eigenvalues have be obtained, they are used to determine the values of the corresponding eigenvectors

$$\tilde{\mathbf{e}}_k = \begin{bmatrix} 1 \\ \tilde{\lambda}_k \\ \dfrac{\tilde{\ }^2 - g \ \tilde{h} + \tilde{B}) + \ \tilde{\lambda}_k - 2\tilde{\ })\tilde{\lambda}_k}{g\tilde{h}} \end{bmatrix}$$

and the wavestrengths

$$\tilde{\alpha}_k = \frac{\tilde{\lambda}_a\tilde{\lambda}_b + g \ \tilde{h} + \tilde{B}) - \tilde{\ }^2)\Delta h + \ 2\tilde{\ } - \tilde{\lambda}_a - \tilde{\lambda}_b)\Delta \quad h) + g\tilde{h}\Delta B}{\tilde{\lambda}_k - \tilde{\lambda}_a) \ \tilde{\lambda}_k - \tilde{\lambda}_b)},$$

where $a \neq k \neq b$. The Roe averages are

$$\tilde{\ } = \frac{\sqrt{h_R} \ {}_{R} + \sqrt{h_L} \ {}_{L}}{\sqrt{h_R} + \sqrt{h_L}}, \quad \tilde{h} = \frac{1}{2} \ h_R + h_L), \quad \tilde{B} = \frac{1}{2} \ B_R + B_L)$$

and

$$\tilde{d} = \frac{\xi\Delta \ A \quad {}^{m})}{\Delta \quad h) - \tilde{\ }\Delta h}.$$

If the Sediment Transport Flux 1.5) is used with $A = 1$ and m is an integer, we can use

$$\tilde{d} = \frac{\xi\ \sqrt{h_R}+\sqrt{h_L})}{\sqrt{h_L}h_R+\sqrt{h_R}h_L}\sum_{k=0}^{m-1} \quad _R)^k \quad _L)^{m-(1+k)}.$$

For the source term approximation, we can use a pointwise approach,

$$\mathbf{R}^n_{+\frac{1}{2}} = \begin{bmatrix} 0 \\ _2\ B^n_{+1}+B^n)\ h^n_{+1}-h^n) \\ 0 \end{bmatrix},$$

or a decomposed approach:

$$\tilde{\beta}_k = \frac{g\tilde{B}\ 2\tilde{}-\tilde{\lambda}_a-\tilde{\lambda}_b)\Delta h}{\tilde{\lambda}_k-\tilde{\lambda}_a)\ \tilde{\lambda}_k-\tilde{\lambda}_b)} \quad \text{where} \quad a \neq k \neq b.$$

3.3 Approximating the Wave Speed of the Bed-Updating Equation

Most of the numerical schemes discussed for **Formulation A-NC** and **Formulation A-CV** require a numerical approximation of the wave speed of the Bed-Updating Equation,

$$\frac{\partial B}{\partial t}+\xi\frac{\partial}{\partial x}=0,$$

whose wave speed is $\lambda = \xi\frac{\partial q}{\partial B}$. Unfortunately, the Sediment Transport Flux, , is not a direct function of B, which can create difficulties in obtaining an accurate approximation of the wave speed.

One approach we can use to approximate the wave speed is to use a finite difference approach,

$$\lambda_{+\frac{1}{2}} = \xi\frac{_{+1} -}{B_{+1}-B} \quad \text{if} \quad B_{+1}-B \neq 0. \qquad 3.11)$$

Unfortunately, the finite difference approach can only be used when $B_{+1} - B \neq 0$ and can also produce an inaccurate approximation of the wave speed when the gradient of the riverbed changes sign. In Section 2, we proved that if we use $\ ) = A\ ^m$ for $\ \geq 0$, then the wave speed of the Bed-Updating

Hence, we can use

$$\lambda_{\ +\frac{1}{2}} = \frac{\xi m}{h^n_{\ +\frac{1}{2}}} [\ \ ^m]^n_{\ +\frac{1}{2}}, \quad \text{where} \quad h^n_{\ +\frac{1}{2}} = \frac{1}{2}\ h^n_{\ +1} + h^n) \qquad 3.12)$$

as an approximation of the wave speed. Unfortunately, the analytical approach can only be used if the Sediment Transport Flux can be written analytically and can only be used for test problems where the height of the riverbed is significantly smaller than the height of the river.

There are numerical schemes that we can use to approximate the Bed-Updating Equation that do not require a value of the wave speed to be calculated. For example, the Lax-Friedrichs scheme,

$$B^{n+1} = \frac{1}{2}\ B^n_{\ +1} + B^n_{\ -1}) - \xi \frac{s}{2}\ \ ^n_{\ +1} - \ ^n_{\ -1}), \qquad 3.13)$$

can be used to approximate the Bed-Updating Equation but the Lax-Friedrichs scheme suffers from dissipation resulting in a very small Courant number being required. Alternatively, since we know that the sign of the wave speed of the Bed-Updating Equation must be positive if we use $\ \ ) = A\ \ ^m$ where $\ \geq 0$, we can use the first order Upwind scheme,

$$B^{n+1} = B^n - \xi s\ \ ^n - \ ^n_{\ -1}), \qquad 3.14)$$

to approximate the Bed-Updating Equation. Alternatively LeVeque & Yee's MacCormack approach 3.4) with $\Phi^{(2)}_{\ +\frac{1}{2}} = 0$ does not require an approximation of the wave speed either and is second order accurate.

4 Numerical Results

In this section, we will discuss

and the bathymetry for

1. **Problem A** is

$$B^* \ x, 0) = \left\{ \sin^2 \left(\pi \ x - 300) \qquad 4300) \right. \right.$$

converging the Shallow ater Equations, we have assumed that the riverbed is fixed. e would expect the pulse in the velocity to decrease for a moving riverbed since the velocity is pushing the pulse in the riverbed resulting in a smaller velocity over the pulse in the riverbed. However, by using **Formulation A-CV**, no spurious oscillations are present in the numerical results even thought we have calculated the Bed-Updating Equation separately. In addition, by using **Formulation A-CV**, we can use the finite difference approach 3.11) to approximate the wave speed since the gradients of the riverbed and the Sediment Transport Flux are never of opposite sign.

For **Formulation A- F** using Roe's Scheme with source term decomposed, Figure 4.11 and Figure 4.12 show that the numerical results do not suffer from spurious oscillations. Surprisingly, even thought the formulation has a singular matrix, the numerical results are still quite accurate.

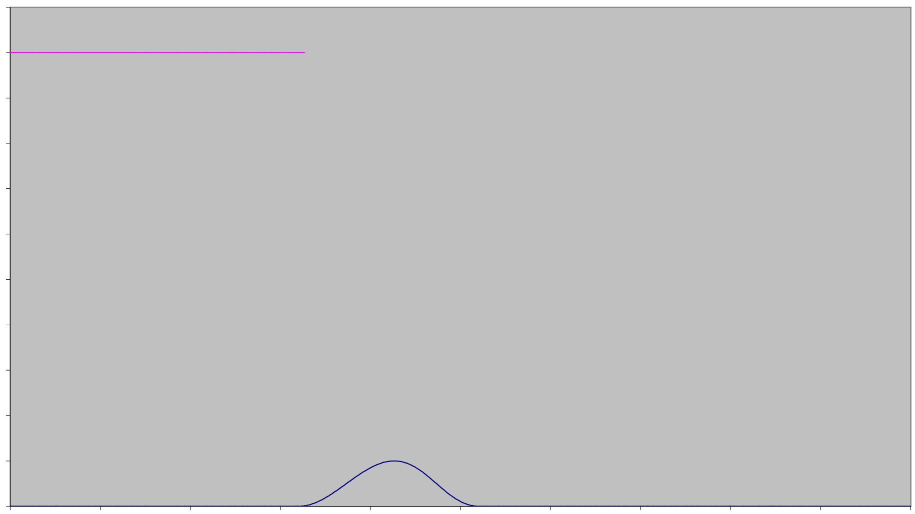
From Figure 4.13 and Figure 4.14, we can see that **Formulation B** using Roe's Scheme has produced very smooth numerical results with no spurious oscillations present.

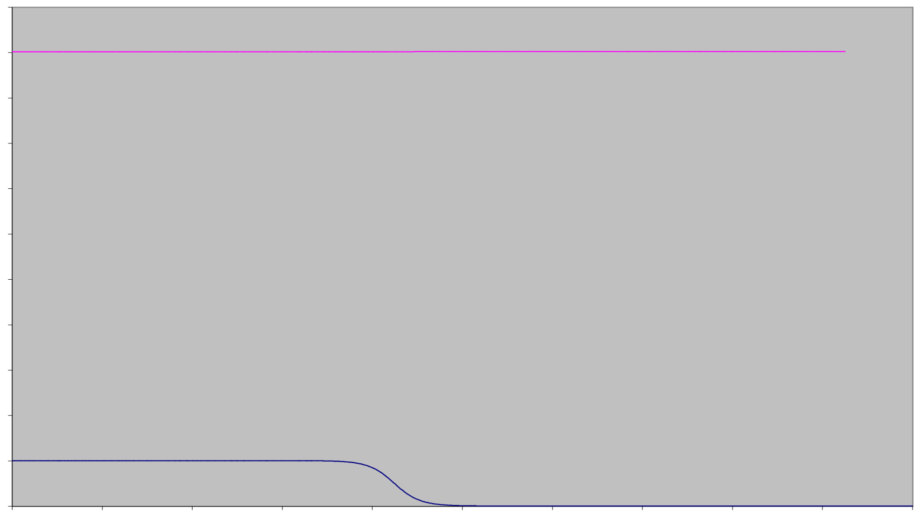
Figure 4.15 and Figure 4.16 show that **Formulation C** using Roe's Scheme with source term decomposed has also produced very smooth numerical results with no spurious oscillations present.

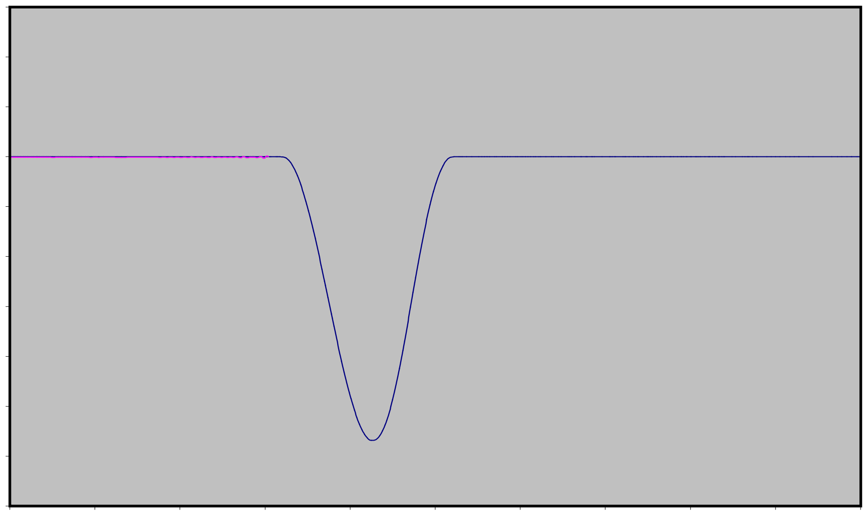
In Figures 4.17 to 4.20, an overall comparison is given at $t = 500s$. Here, we can see that the most accurate formulation was **Formulation B**, closely followed by **Formulation C**. **Formulation A- F** produced quite accurate numerical results but differed slightly to **Formulation B** and **Formulation C**. **Formulation A-NC** gave the least accurate numerical results as far as spurious oscillations is concerned but **Formulation A-CV** moved the pulse in the riverbed at a completely different wave speed than the other approaches. Hence, converging the Shallow ater Equations to a steady state solution before updating the riverbed is incorrect as the approach assumes that the riverbed is fixed.

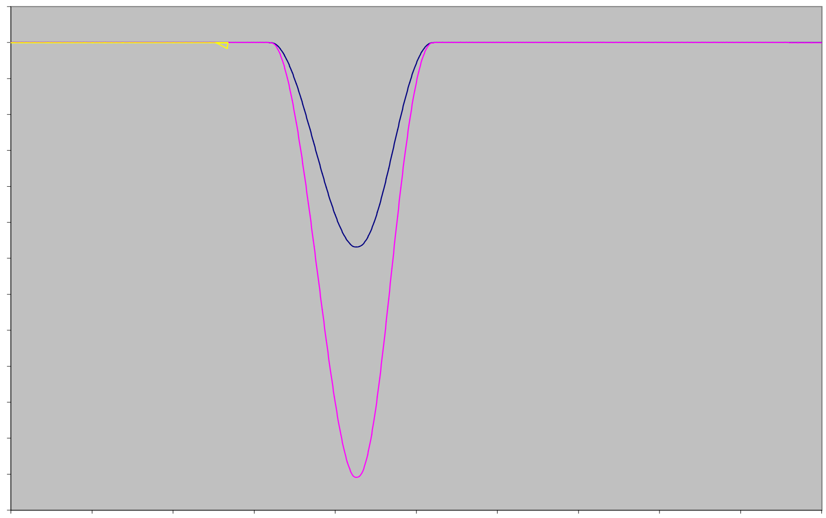
4.2.2 LeVeque & Yee's MacCormack Approach

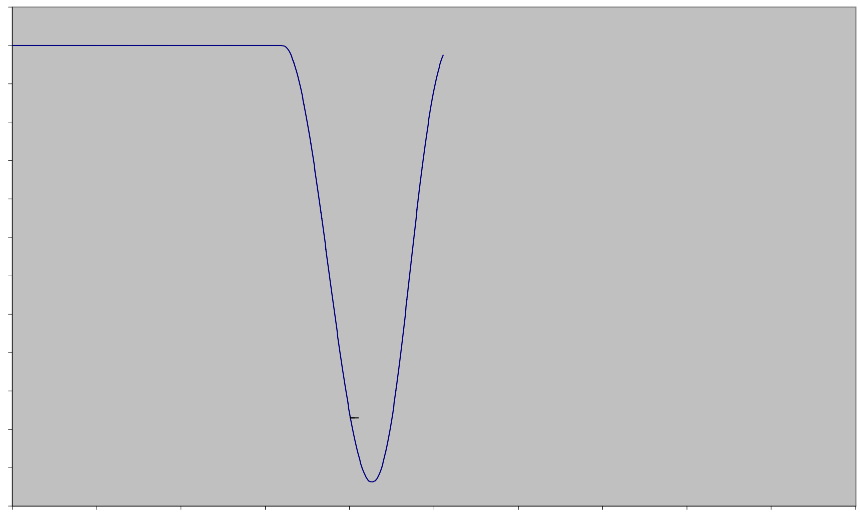
For LeVeque & Yee's MacCormack approach, an overall comparison of the different formulations is given at $t = 500s$ in Figures 4.21 to 4.24. Here, we can see that **Formulation A-CV** has again moved the pulse in the riverbed at a completely different wave speed than the other approaches. However, for the MacCormack approach, **Formulation A- F** has produced spurious oscillations that have overpowered the numerical solution. These spurious oscillations cannot be minimised using a smaller Courant number. By using the second order version of LeVeque & Yee's MacCormack approach, the spurious oscillations are considerably smaller but we require a numerical scheme that satisfies the TVD property. **Formulation A-NC** has not produced spurious oscillations for LeVeque & Yee's MacCormack approach but did for Roe's Scheme with source term decomposed. However, by looking at Figure 4.21, we can see that **Formulation A-NC** has produced slight waves on the right side of the pulse and the maximum height of the pulse differs from the other formulations. As with Roe's Scheme with source term decomposed, **Formulation B** and **Formulation C** have produced the most accurate numerical results for the different using Roe's Scheme. As with **Problem A**, the numerical results obtained are very smooth and do not suffer from spurious

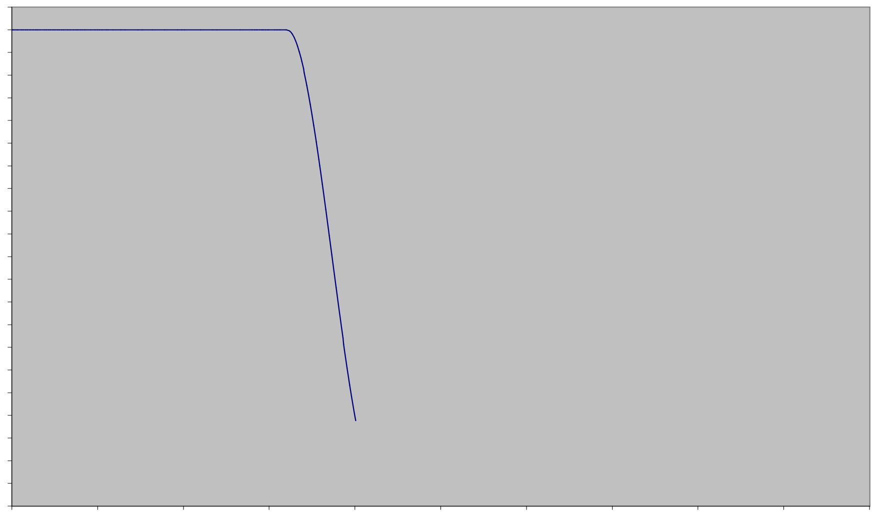


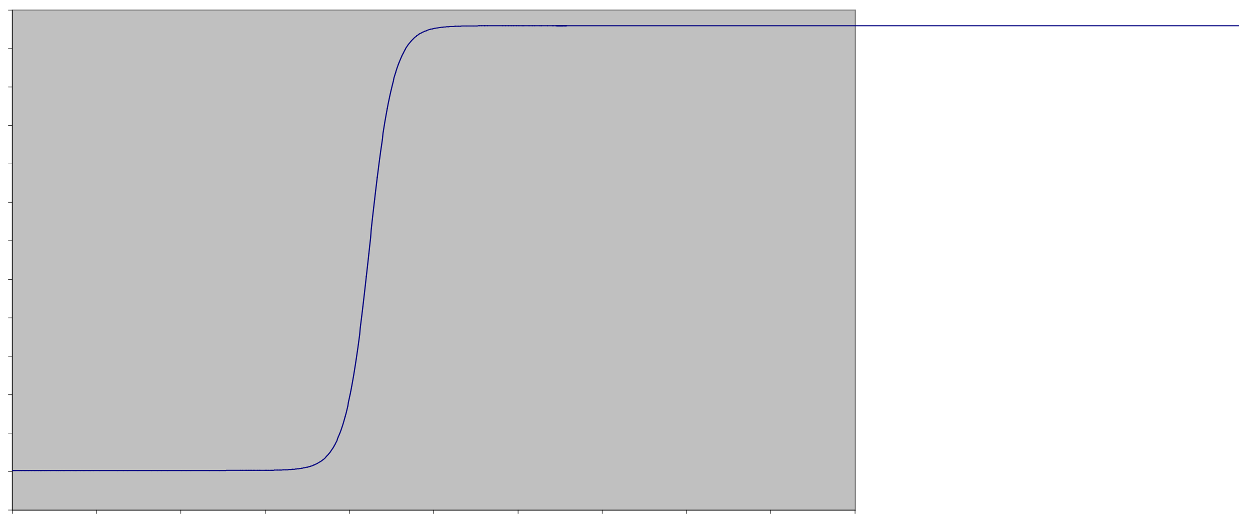


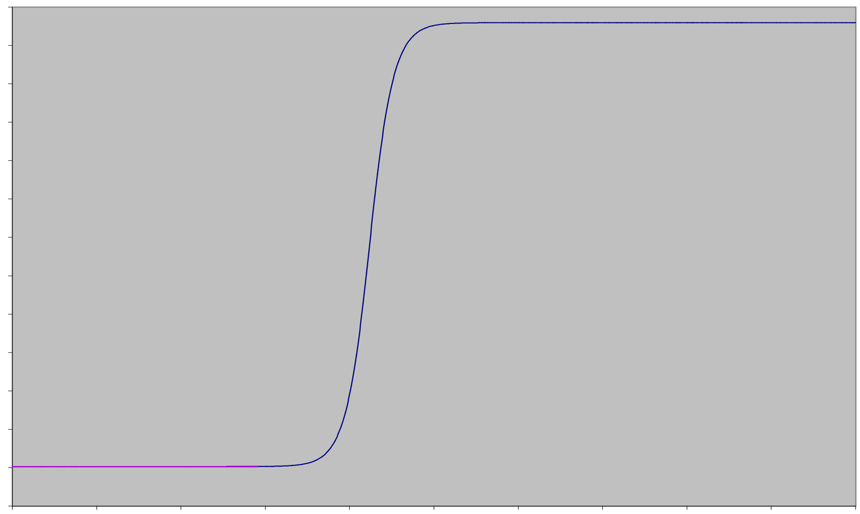


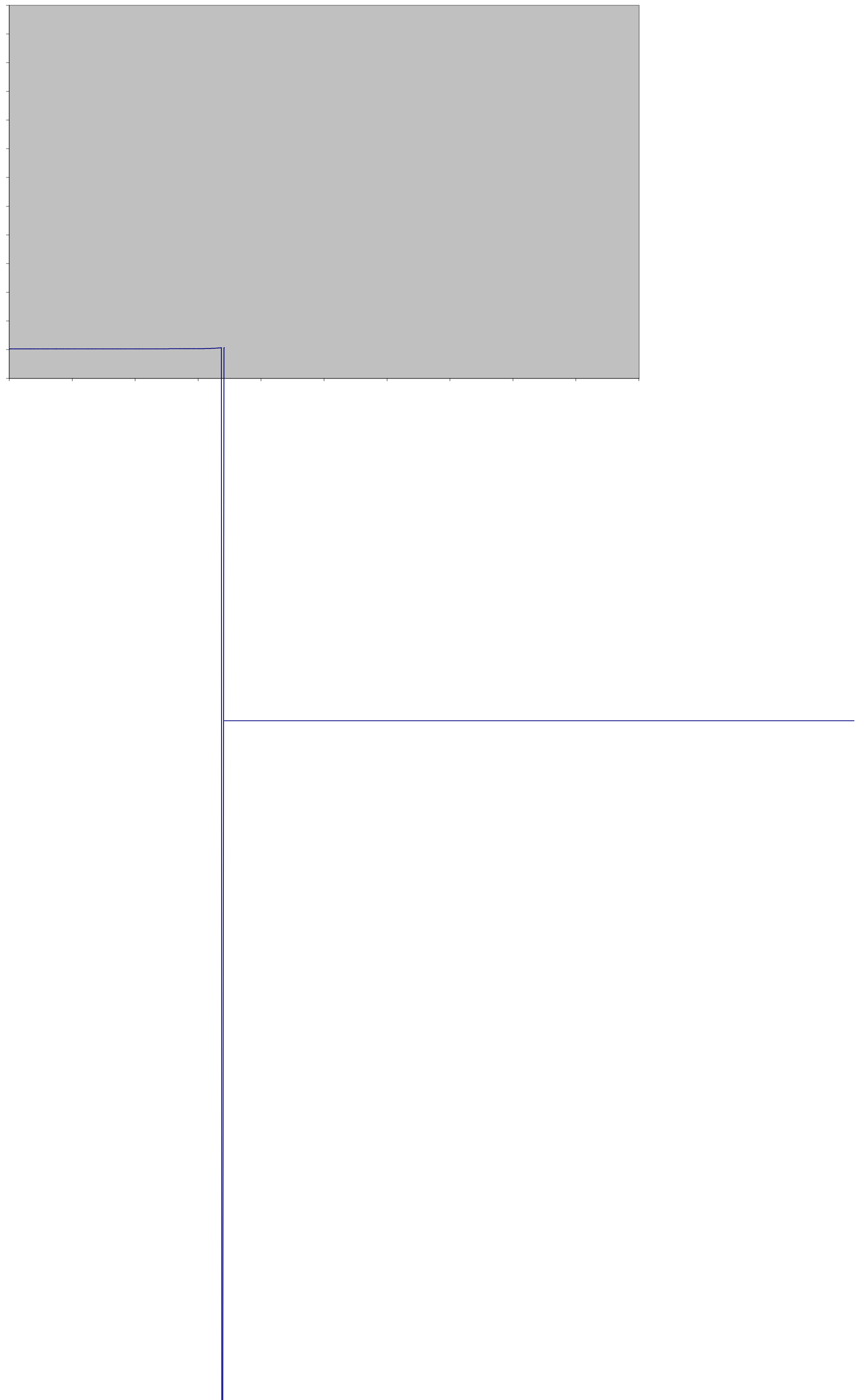


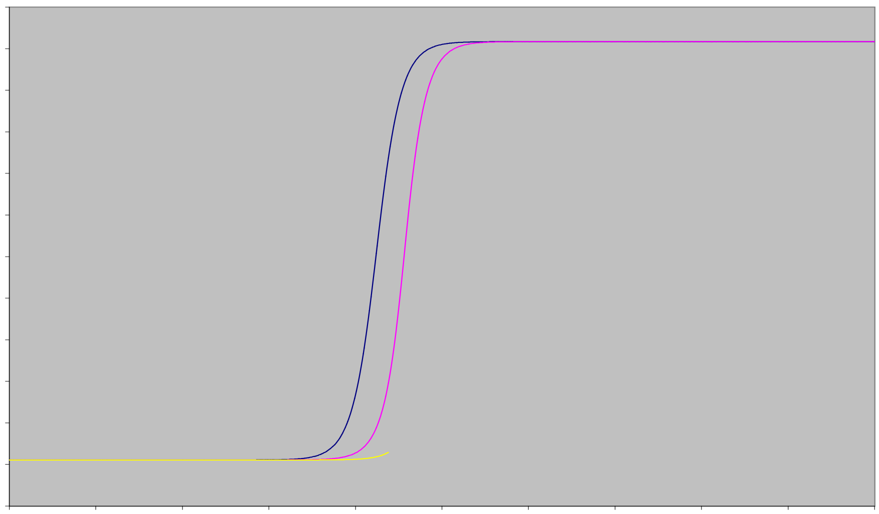


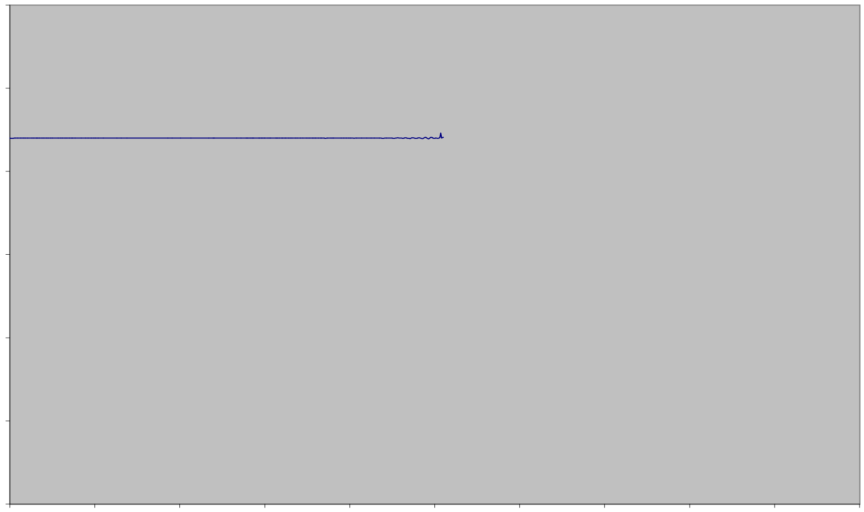


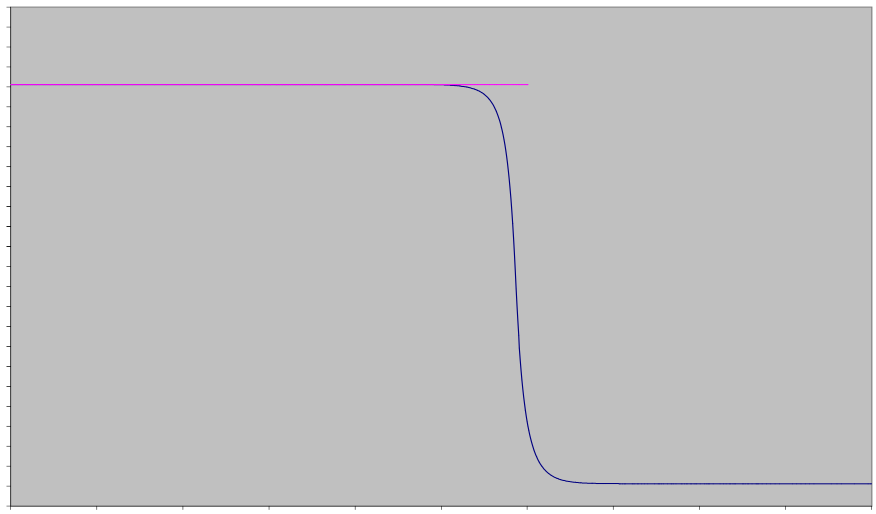












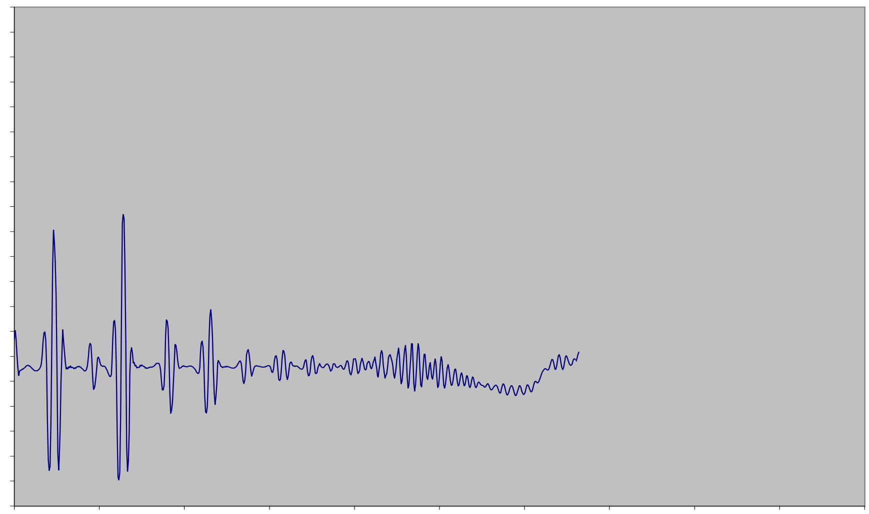
5 onclusion

Throughout this report, we have

5.1 to 5.2. However, by using a staggered approach,

$$
{}^{*}_{+\frac{1}{2}} = \begin{cases} {}^{n+1} + \frac{1}{2}\ 1 - \nu^{n+1}_{+\frac{1}{2}})\ \ {}^{n+1}_{+1} - \ {}^{n+1})\Phi^{n} & \text{if } \nu^{n+1}_{+\frac{1}{2}} > 0 \\ {}^{n+1}_{+1} - \frac{1}{2}\ 1 + \nu^{n+1}_{+\frac{1}{2}})\ \ {}^{n+1}_{+1} - \ {}^{n+1})\Phi^{n} & \text{if } \nu^{n+1}_{+\frac{1}{2}} < 0 \end{cases},
$$

we can minimise these spurious oscillations without having to reduce the Courant number, see Figure 5.1 and Figure 5.2.



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