On the accuracy of One-Dimensional Models of Steady Converging/Diverging Open Channel Flows. *

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Abstract

Shallow wat r flows through op n chann ls with varying br adth ar commonly mod ll d by a syst m of on -dim nsional quations, d spit th two-dim nsional natur of th g om try and th solution. In this work st ady stat flows in conv rging/div rging chann ls ar studi d, in ord r to d t rmin th rang of param t rs (flow sp d and chann l br adth) for which th assumption of quasi-on -dim nsional flow is valid. This is don by comparing both xact and num rical solutions of th on -dim nsional mod l with num rical solutions of th corr sponding two-dim nsional flows, and it is shown that v n for appar ntly g ntl constrictions, for which th assumptions from which th on -dim nsional mod l is d riv d ar valid, significant diff r nc s can occur. Furth rmor , it is shown how th natur of th flow d p nds on th mann r in which th boundary conditions ar appli d by contrasting th solutions obtain d from two commonly us d approach s.

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1 Introduction

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2 The one-dimension l model

In one dimension, shallow water flow through an open channel of rectangular cross-section and variable breadth can be modelled by the equations

$$\begin{pmatrix} Bd \\ Bdu \end{pmatrix}_t + \begin{pmatrix} Bdu \\ Bdu^2 + \frac{1}{2}gBd^2 \end{pmatrix}_x = \begin{pmatrix} 0 \\ \frac{1}{2}gd^2 B_x \end{pmatrix}, \tag{2.1}$$

in which d represents the depth of the flow, u is its velocity, $B = B(x)$ is the variable breadth of the channel and g is the acceleration due to gravity (see, for example [2] for their derivation). Essentially, Equation (2.1) can be derived from the more general, two-dimensional shallow water model under the assumption that $B_x = O(\epsilon)$ for $\epsilon \ll 1$, so that the transverse acceleration of the flow is negligible in comparison with the longitudinal acceleration. In these circumstances the variables d and u are considered to be breadth-averaged quantities. Only steady state solutions are considered in this work, for which the time derivatives are zero. They are only included in (2.1) because they are often used as a numerical device to converge to steady solutions, as is done in the schemes of Section 4 which provide the approximate solutions.

Exact steady state solutions of (2.1) are simple to construct (see for example [7] and related work in which the source terms represent variable bed topography [4, 1]). For a converging/diverging channel with continuously varying breadth the steady solutions of (2.1) can be divided into four categories:

A. continuous - purely subcritical (possibly critical at the most narrow point of the channel, the throat).

B. discontinuous - subcritical at inflow, passing smoothly to supercritical at the throat, then back to subcritical via a stationary hydraulic jump in the diverging region of the channel, remaining so until outflow.

C. continuous - subcritical at inflow, passing smoothly to supercritical at the throat of the channel, and remaining supercritical to outflow.

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The particular form taken by the steady solution depends on the boundary conditions which are applied at the entrance and the exit of the channel section being modelled.

The simplest cases are A and D. Integration of the steady equations leads straightforwardly to two quantities which remain constant throughout the channel. These are the total discharge

$$Q \;=\; Bdu\,, \tag{2.2}$$

and the total head

$$H_{\mathrm{T}} \;=\; \frac{u^2}{2g} + d \;=\; \frac{Q^2}{2gB^2d^2} + d\,. \tag{2.3}$$

Given that values for Q and H_{T} can be deduced from the boundary conditions, combining (2.2) with (2.3) leads to

$$d^3 - H_{\mathrm{T}}d^2 + \frac{Q^2}{2gB^2{H_{\mathrm{T}}}^3} \;=\; 0\,, \tag{2.4}$$

an algebraic equation relating the depth of the flow d to the local channel breadth B. This has a pair of physically admissible (positive) solutions for d, one representing subcritical flow and the other supercritical flow, on condition that

$$\frac{B}{B_{\mathrm{in}}} \;\geq\; F_{\mathrm{in}} \left(\frac{3}{F_{\mathrm{in}}^2 + 2} \right.^{\frac{3}{2}} \tag{2.5}$$

for all values of B in the given channel geometry, where $F_{\mathrm{in}} = u_{\mathrm{in}}/\sqrt{gd_{\mathrm{in}}}$ is the local Froude number specified at inflow. The solution which is chosen by the equations (2.1) depends on the boundary conditions applied (unless both Q and H_{T} are specified, in which case the choice remains open).

When equality holds in (2.5) for some value of B within the channel geometry the flow becomes critical for the values of Q and H_{T} implied by the boundary conditions. However, any critical point of the flow must lie at the throat of the channel so, unless equality is satisfied there, the inlet values of Q and/or H_{T} change automatically to satisfy the boundary condition at inflow *and* the critical condition at the throat ($F = 1$ when $B = B_{\mathrm{min}}$). The flow is then of type B or C. Furthermore, the variation of the Froude number upstream of the critical point in such situations is uniquely defined, being the subcritical solution ($0 < F < 1$) of the equation

$$F^2 \left(\quad \frac{3}{3} \right.$$

This implies that the Froude number at inflow is fixed by the channel geometry, taking the same value whenever the solution is transcritical (so F is not a practical choice for specification as an inflow boundary condition). The new values of Q and H_{T} for the smooth region of the flow which surrounds the critical point can be calculated by combining the subcritical inflow boundary condition with this inflow Froude number.

Downstream of the critical point, the flow type (B or C) is determined by the outflow boundary conditions. Initially, since the flow is continuous through the critical point, the solution retains the upstream values of Q and H_{T} but switches to the supercritical branch of (2.4) downstream of the throat. If no jump occurs the supercritical solution values found using (2.4) are retained throughout the rest of the channel.

When a stationary hydraulic jump occurs (which must always be from supercritical flow to subcritical flow), equations (2.1) lead to two quantities which are continuous across the jump. These are given by

$$[du] \;=\; 0 \qquad \text{and} \qquad \left[du^2 + \frac{1}{2}gd^2\right] \;=\; 0\,. \tag{2.7}$$

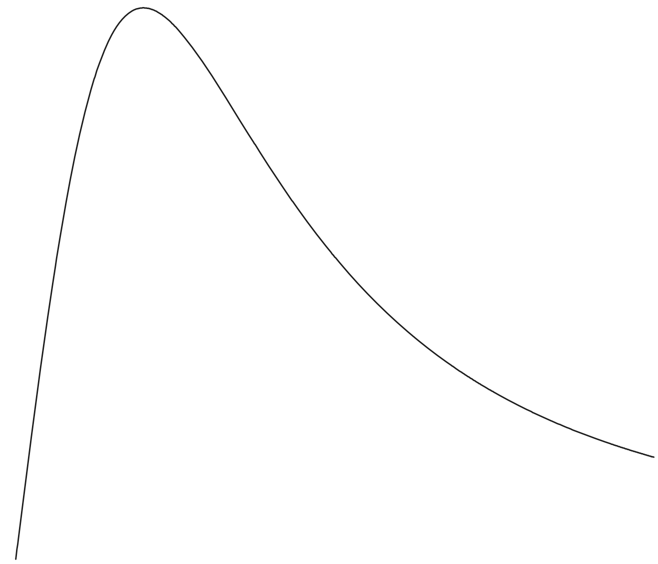
The first of these, together with (2.2), implies that Q is constant throughout the domain for any steady flow, but from the second and (2.3) it is clear that there is a jump in H_{T} when the flow is discontinuous. Thus the flow downstream of a stationary hydraulic jump is determined by the value of Q which has been calculated for the transcritical upstream flow and the boundary condition specified at outflow.

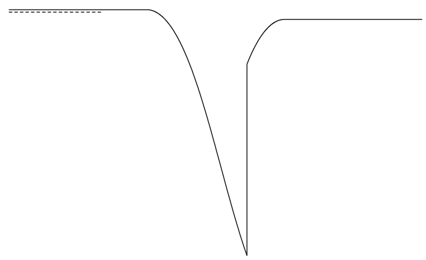
Combining the two expressions (2.7) leads to a relationship between the branches of the solution on either side of the jump, given by

$$d_+ \;=\; \frac{d_-}{2}\left(\sqrt{1+8F_-{}^2}-1\right)\,, \tag{2.8}$$

in which d_+ is the depth immediately downstream of a discontinuity, while d_- and F_- are the depth and local Froude number immediately upstream. The flow sustains a stationary hydraulic jump if

$$(d_+)_{\text{out}} \;\le\; d_{\text{out}} \;\le\; d_{\text{in}} \tag{2.9}$$





4 Numeric l results

The one-dimensional numerical scheme used in this work combines Roe's approximate Riemann solver [11], as applied to the shallow water equations (2.1) [10], with the minmod limiter [12] within a MUSCL algorithm [13], together with a recently developed upwind discretisation of the source term [6]. In two dimensions, the discretisation used is the multidimensional upwind method of Mesaros and Roe [3, 8], applied to the shallow water equations, (3.1) and (3.2), on unstructured triangular grids [5].

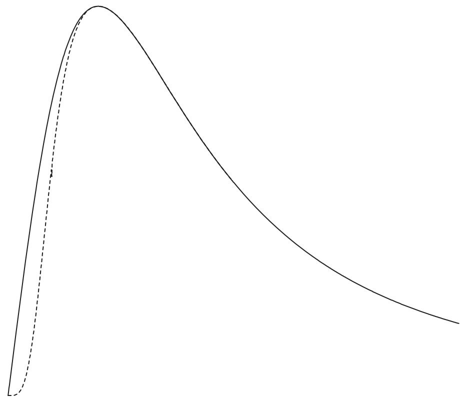
For the purposes of this comparison, each of the results presented is for a channel of length 3 units which has a symmetric constriction of length 1 unit at its centre whose breadth is given by

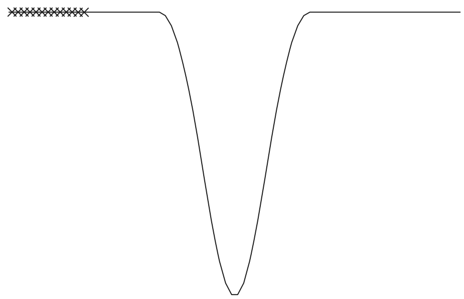
$$B(x) = \begin{cases} 1.0 - (1.0 - B_{\min})\cos^2(\pi(x-1.5)) & \text{for} \quad |x-1.5| \leq 0.5 \\ 1.0 & \text{otherwise}, \end{cases} \tag{4.1}$$

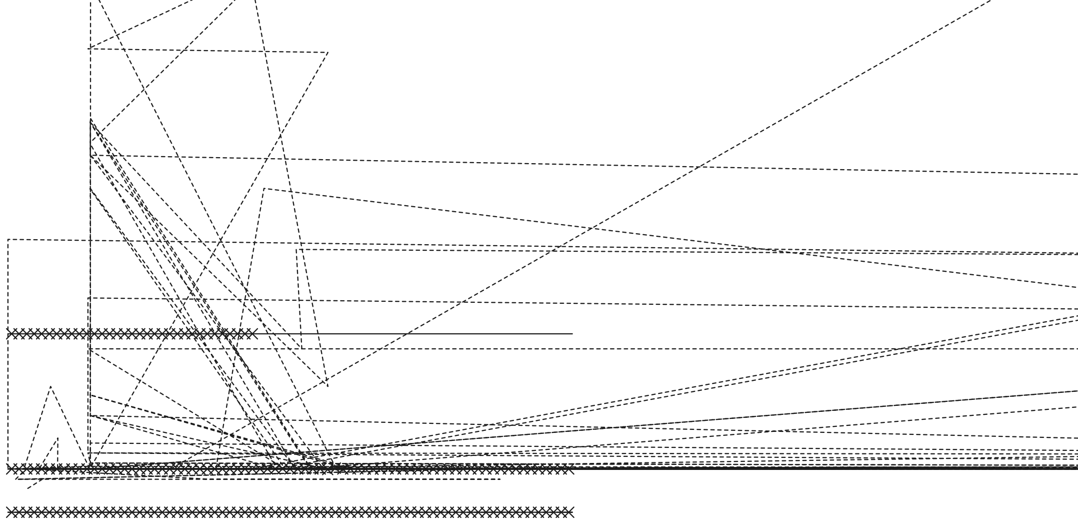
where $B_{\min}$ is the minimum channel breadth and x is the distance into the channel (so the throat is positioned at the midpoint of the constriction). In the two-dimensional case the constriction has been chosen for simplicity to be represented by symmetric indentations on either side of the channel (as illustrated in Figure 4.6). Whilst alternative constructions undoubtedly alter the flow in some way, their effect on the comparison with one-dimensional results is not significant.

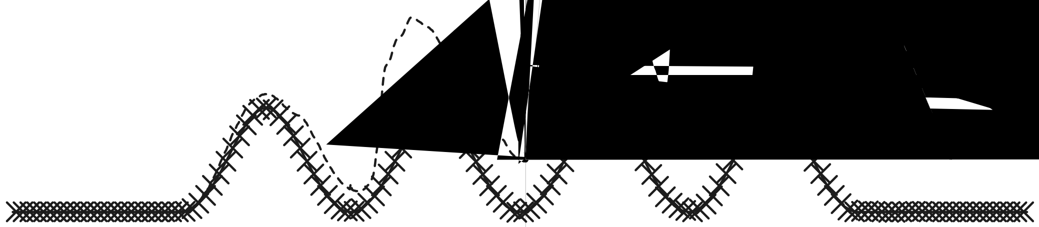
Each of the one-dimensional numerical solutions is obtained on a uniform 76 node grid, giving comparable resolution to the two-dimensional grids used, each of which has been constructed using a simple advancing front technique (see for example [9]) with an underlying mesh spacing parameter of 0.04. The initial conditions for each numerical experiment (in which the steady state solution is achieved by approximating the evolution of the time-dependent shallow water equations (2.1) with steady boundary conditions and converging to the steady state from the initial conditions as $t \to \infty$) were $d = 1.0$ and $F = F_{\text{in}}$, with $v = 0.0$ in two dimensions.

Figure 4.1 shows how well the one-dimensional numerical results agree with the theory (as illustrated in Figure 2.1) in terms of the parameter values ($B_{\min}$









channel with a quadruple symmetric constriction with $B_{\min} = 0.9$ and $F_{\text{in}} = 1.9$. The one-dimensional model predicts smooth supercritical flow throughout, but the comparison with two dimensions becomes progressively worse as the jumps interact with each other.

5 Conclusions

In this work a comparison has been made between one- and two-dimensional models of steady state shallow water flow through an open channel of varying breadth. It has been shown that the numerical and analytical solutions to the one-dimensional model agree closely, provided that an appropriate discretisation of the source terms is employed.

When the flow is completely smooth and subcritical these solutions also prove to be an accurate prediction of the breadth-averaged two-dimensional flow. For small constrictions ($B_x \ll 1$) the agreement remains good even when the one-dimensional model predicts a discontinuous flow, because the transverse acceleration in the flow is negligible and consequently the two-dimensional solution remains essentially one-dimensional. As the constriction narrows, however, steady hydraulic jumps become more curved and the one-dimensional model less accurate. When the flow downstream of the constriction is supercritical, the undular jumps which are propagated from the constriction in the two-dimensional case cannot be predicted by the one-dimensional equaagreemeaTf2TD2Tc2-menusrtheedccu-TJ2yTD to be more robust.

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